

# Particle production in string cosmology models

Ram Brustein

*Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel, and  
Theory Division, CERN, CH-1211, Geneva 23, Switzerland  
email: ramyb@bgumail.bgu.ac.il*

Merav Hadad

*School of Physics and Astronomy, Beverly and Raymond Sackler Faculty of Exact Sciences,  
Tel Aviv University, Tel Aviv 69978, Israel  
email: meravv@post.tau.ac.il*

We compute spectra of particles produced during a dilaton-driven kinetic inflation phase within string cosmology models. The resulting spectra depend on the parameters of the model and on the type of particle and are quite varied, some increasing and some decreasing with frequency. We use an approximation scheme in which all spectra can be expressed in a nice symmetric form, perhaps hinting at a deeper symmetry of the underlying physics. Our results may serve as a starting point for detailed studies of relic abundances, dark matter candidates, and possible sources of large scale anisotropy.

Preprint Numbers: BGU-PH-97/12, TAUP-2448-97

## I. INTRODUCTION

An inflationary scenario [1,2] (the so called “pre-big-bang” scenario) postulates that the evolution of the Universe starts from a state of very small curvature and coupling and then undergoes a long phase of dilaton-driven kinetic inflation, in which the curvature and coupling grow, and then at some later time joins smoothly standard Friedman-Robertson-Walker (FRW) radiation dominated cosmological evolution, thus giving rise to a singularity free inflationary cosmology. Since the graceful exit transition from the dilaton-driven phase to the decelerated FRW evolution cannot occur while curvatures and coupling are small [3,4], an intermediate high curvature “string phase” [5], in effect replacing the big bang, has to separate the early inflationary phase from the late FRW phase. The required initial conditions of the Universe in this scenario were discussed [6–8], and the graceful exit transition is now better understood [9,10]. We assume that the appropriate initial conditions were chosen such that a long dilaton-driven inflationary phase was indeed part of the evolution and that a later standard FRW cosmology followed, and wish to study possible consequences.

We study production of a variety of particle types during the inflationary dilaton-driven phase. So far, most of the produced particle spectra were found to rise sharply with frequency. Therefore, in these spectra, most of the power is concentrated at higher frequencies, a property with interesting consequences [5,11–13], but also some disadvantages. In particular, at very large wavelengths there is almost no power, making these inhomogeneity perturbations an unlikely source for large scale anisotropy observed in the cosmic microwave background and required as seeds for structure formation. The standard explanation given for the generic spectral frequency dependence is that since the curvature increases sharply, particle production also increases and hence the resulting spectrum. Certain axionic perturbations were so far the only exception found to this tendency [14]. We find that although sharply rising spectra are indeed common, flatter spectra or even decreasing spectra are just as likely. We find that the slope of the spectrum depends on the spin of the particle and on the type of dilaton prefactor and whether it is massless or massive, revealing a rich range of spectral shapes, of which many more deserve further individual attention.

The model of background evolution we adopt in this paper is a simplified model. The evolution of the Universe is divided into four distinct phases, the first phase is a long dilaton-driven inflationary phase. We assume that the background solution is the simplest solution of the string-dilaton-gravity equations of motion in 4 dimensions, the so called (+) branch vacuum. The second phase is a high-curvature string phase of otherwise unknown properties, followed by ordinary FRW radiation dominated (RD) evolution and then a standard FRW matter dominated (MD) evolution, both with a fixed dilaton. We assume that curvature stays high during the string phase, in a sense that we define better later, but we do not assume any specific form of background evolution. Obviously, by doing that we give up on the possibility of obtaining any information about the produced particles during the string phase. We postpone such analysis until some better ideas and more reliable methods for handling the string phase are at hand

(see, however, [5,15]). We do not lose the ability to compute the spectra produced during the dilaton-driven phase, as we explain later. We parametrized our ignorance about the string phase background, as in [5], by the ratios of the scale factor and the string coupling, at moments in (conformal) time marking the beginning and end of the string phase  $z_S = a_1/a_S$  and  $g_1/g_S$ . It turns out that in the generalized setting we use, these two parameters are still sufficient to parametrize all spectra.

Particles get produced during the dilaton-driven phase by the standard mechanism of amplification of vacuum fluctuations [16,17]. Deviations from homogeneity and isotropy are generated by quantum fluctuations around the homogeneous and isotropic background and then amplified by the accelerated expansion of the Universe. In practice, we compute particle production in the standard way, by solving linearized perturbation equations with vacuum fluctuations boundary conditions. The general solution of the perturbation equation is a linear combination of two modes, one which is approximately constant for wavelengths large compared with the curvature of the Universe (outside the horizon), and one which is generically time dependent outside the horizon. We understand the appearance of a constant mode as the freezing of the perturbation amplitude, since local physics is no longer active on such scales. The existence of the time dependent mode can be most easily understood in terms of a constant mode of the conjugate momentum of the perturbation [18]. The amplitude of the conjugate momentum also freezes outside the horizon, since local physics is no longer active on such scales. This forces the time derivative of the perturbation to be non-vanishing, leading to a “kinematical” time dependence of the perturbation amplitude. The analysis of perturbation equations and their solutions in the Hamiltonian formalism was first suggested in [18], as a way of understanding certain symmetries of the spectra of gravitons and photons.

In addition to the dilaton and metric, string theory contains many other fields that have trivial expectation values in our model of background evolution and do not affect the classical solutions, but they do fluctuate. We are interested in “low-mass” fields, which are either massless, or have masses much below the string scale, such as moduli, gauge bosons, and their superpartners. In realistic models the mass of different fields may depend in a complicated way on the dilaton or other moduli. We will take the mass as an additional parameter and assume that it vanishes in the dilaton-driven phase and takes a constant value, much smaller than the string mass  $M_s = M_p g_1$  ( $M_p$  is the Planck mass), from the start of the RD era and on. This assumption could be relaxed without substantial changes in results. For simplicity, we assume that the produced particles are stable particles, decoupled from the radiation thermal bath. Adding the effects of interactions with the thermal bath and decay is rather straightforward and has to be done on a case by case basis.

For a large class of particles, it is enough to specify a particle type with two parameters,  $m$  and  $l$ , specifying the interaction of the field with the background metric and dilaton. We explain the appearance of these parameters in more detail later, but for now we give the following list for some particular cases. For the dilaton or graviton  $m = 1$  and  $l = -1$ , for the model independent axion  $m = l = 1$ , for perturbative gauge fields such as photons,  $m = 0$  and  $l = -1$ , while for some non-perturbative gauge fields  $m = 0$  and  $l = 0$ , Ramond-Ramond axions and non-perturbative scalars have  $m = 1$  and  $l = 0$ . In general, fields with more tensor indices tend to have smaller (more negative)  $m$ ’s. We do not know at the moment a general rule for allowed values of  $m$  and  $l$ , but it is obvious that many combinations are possible.

Possible benefits of our results are expected to come from several different aspects. First, from the different dependence on the fundamental parameters  $g_S$ ,  $z_S$  of decreasing as well as increasing spectra. This leads to different constraints on the allowed range of parameters, allowing to narrow their range and give better prediction for the spectrum of relic gravitational radiation background. Flatter spectra allow the possibility of producing large scale inhomogeneities with enough power to explain observed cosmic microwave background anisotropies. Another exciting possibility is that some of the weakly interacting particles will constitute the dark matter in the Universe. Our results provide a starting point for actual calculations regarding these issues and we hope to study them in the future.

Until now, the spectra that were computed were the following. Production of gravitons, for which the full dependence on parameters, effects of late time evolution and observability were taken into account [5,11]. Production of dilatons was partially studied [13], not taking into account the effects of an intermediate string phase. Photon production was studied [12], taking into account dependence on parameters and late time evolution. Certain axionic perturbations were studied without taking into account the effects of an intermediate string phase and the effects of mass [14]. In [19] some additional fields were studied.

Because this work is the first attempt to make a comprehensive survey of particle production in string cosmology models, in the hope of discovering some interesting phenomena, as we indeed do, we adopt a modest, minimalist approach. Our main goal in this paper is to show that spectra of produced particles come in different shapes, and we feel that this point is better demonstrated with spectra than can be reliably computed. We use approximations that we judge not to hide our main results and are accurate enough for our purposes. There are, however, many obvious improvements, which could be implemented if necessary. We would like to issue a warning about numerical coefficients, which should not be taken too seriously, in view of the approximations we used. In the same spirit we make no attempt to obtain bounds from constraints on the total energy in fluctuations, or to require that the total

number of particles leads to the observed radiation (as in [19]), or to impose any other additional requirements on the spectra. We believe that this should be done for each class of particles separately, with additional input in the form of assumptions about stability, interactions and dependence on late stages of the evolution. We are completing a study of relic abundance axions [20], and plan to study some additional interesting cases.

The plan of the paper is as follows. In section 2 we describe in more detail the model of background evolution, the perturbation equations, and their boundary conditions. In section 3 we describe our method of solving the perturbation equations, and solve them. The paper is quite technical and we have made a special effort to summarize our results in a self-contained form in section 4. The last section contains our conclusions and some preliminary consequences.

## II. BACKGROUND EVOLUTION AND PARTICLE PRODUCTION

In this section we describe the model of background evolution, the perturbation equations obeyed by the different fields, and our method of solving these equations.

### A. The model of background evolution

The model of background evolution we adopt in this paper is a simplified model. The evolution of the Universe is divided into four distinct phases, the first phase is a long dilaton-driven inflationary phase, the second phase is a high-curvature string phase of otherwise unknown properties, followed by ordinary FRW RD evolution and then a standard FRW MD evolution. We assume throughout an isotropic and homogeneous four dimensional flat Universe, described by a FRW metric with the line element  $ds^2 = a^2(\eta) (d\eta^2 - \delta_{ij} dx^i dx^j)$ , where  $\eta$  is conformal time and  $a(\eta)$  is the scale factor. The dilaton  $\phi$  is time dependent,  $\phi = \phi(\eta)$ . All other fields that we will be interested in are assumed to have a trivial VEVs. Therefore, to specify the phases in a concrete way, we need to specify two functions,  $a(\eta)$  and  $\phi(\eta)$  and the conformal time boundaries between the phases, which we proceed to do.

- The dilaton-driven inflationary phase lasts while  $\eta < \eta_s$ , and while it lasts the scale factor and dilaton are given by the solution of the lowest order string-dilaton-gravity equations of motion, the so-called (+) branch vacuum,

$$a = a_s \left( \frac{\eta}{\eta_s} \right)^{(1-\sqrt{3})/2}, \quad e^\phi = e^{\phi_s} \left( \frac{\eta}{\eta_s} \right)^{-\sqrt{3}} \quad (2.1)$$

where  $a_s = a(\eta_s)$  and  $\phi_s = \phi(\eta_s)$ . Both curvature and coupling are growing in this phase, which is expected to last until the time  $\eta_s$  when the curvature reaches the string scale and the background solution starts to deviate substantially from the lowest order solution. For recent ideas about how this may come about see [9,10].

- The string phase lasts while  $\eta_s < \eta < \eta_1$ . We assume that curvature stays high during the string phase. As in [21], we assume that the string phase ends when  $H(\eta_1)/a(\eta_1) \simeq M_s$ . We therefore implicitly assume that the string coupling at the end of the string phase is higher than it's value at the beginning of the string phase and that it does not decrease much during the string phase. But we do not assume any specific form of background evolution. Obviously, by doing that we give up on the possibility of obtaining any information about the produced particles during the string phase. We postpone such analysis until some better ideas and more reliable methods for handling the string phase are at hand (see, however, [15]). We do not lose the ability to compute the spectra produced during the dilaton-driven phase, as we explain later. We parametrized our ignorance about the string phase background, as in [5], by the ratios of the scale factor and the string coupling  $g(\eta) = e^{\phi(\eta)/2}$ , at the beginning and end of the string phase  $z_S = a_1/a_S$  and  $g_1/g_S$ , where  $g_1 = e^{\phi(\eta_1)/2}$  and  $g_S = e^{\phi(\eta_S)/2}$ . It turns out that also in the generalized setting we use, these two parameters are enough. We take the parameters to be in a range we consider reasonable, for example,  $z_S$  could be in the range  $1 < z_S < e^{45} \sim 10^{20}$ , to allow a large part of the observed Universe to originate in dilaton-driven phase, and  $g_1/g_S > 1$ . Note that unlike [5] or [15] we do not assume a specific background during the string phase. The time  $\eta_1$  marks the start of the RD era as we explain below. Some other useful quantities that we will need are  $\omega_1$ , the frequency today, corresponding to the end of the string phase, estimated in [21], and the frequency  $\omega_s = \omega_1/z_S$ , the frequency today corresponding to the end of the dilaton-driven phase.

Our definition of the string phase encompasses the whole duration in between the end of the dilaton-driven phase and the onset of RD phase, which may be quite complicated as in [10].

- The FRW RD phase is assumed to follow the string phase and last while  $\eta_1 < \eta < \eta_{eq}$ ,

$$a = a_1 \left( \frac{\eta}{\eta_1} \right), \quad e^\phi = e^{\phi_1} = \text{const.} \quad (2.2)$$

where  $\eta_{eq}$  is the time of matter-radiation equality,  $a_1 = a(\eta_1)$  and  $\phi_1 = \phi(\eta_1)$ . Note that the dilaton is taken to be strictly constant, frozen at its value at  $\eta = \eta_1$ .

- The FRW MD phase is assumed to last while  $\eta_{eq} < \eta < \eta_0$

$$a = a_{eq} \left( \frac{\eta}{\eta_{eq}} \right)^2, \quad e^\phi = e^{\phi_1} = \text{const.} \quad (2.3)$$

where  $\eta_0$  is today's conformal time and  $a_{eq} = a(\eta_{eq})$ . Note that the constant value of the dilaton is further assumed to be that of today. Some possible consequences of late time evolution of the dilaton were discussed in [22].

For more accurate results, we could have used a more sophisticated description of the RD and MD phases using entropy conservation, effective number of degrees of freedom and temperature, as was done in [21].

We presented our model in the string frame, using string frame conformal time. Another favorite choice is the Einstein frame, in which the gravitational action takes the standard Einstein form. We continue to use the string frame exclusively throughout the paper. General arguments [23], and many explicit examples have shown that physical quantities such as energy density or number density can be computed in any frame.

## B. The perturbation equation

Deviations from homogeneity and isotropy are generated by quantum fluctuations around the homogeneous and isotropic background and then amplified by the accelerated expansion of the Universe. In addition to the dilaton and metric, string models contain many other fields that have trivial expectation values in our model of background evolution and do not affect the classical solutions, but they do fluctuate. We are interested in “low-mass” fields, which are either massless, or have masses much below the string scale, such as moduli, gauge bosons, and their superpartners.

The action for each field's perturbation is obtained by expanding the 4 dimensional effective action of strings, which generically, for a tensor field of rank  $N$ , has the form  $\frac{1}{2} \int d^4x \sqrt{-g} e^{l\phi} \mathcal{L}^2(T_{\mu_1, \dots, \mu_N})$ , where the parameter  $l$  in the dilaton prefactor is determined by the type of field. The second order covariant Lagrangian  $\mathcal{L}^2$  contains a kinetic term and possibly also a mass term. Setting the dilaton and metric at their background values,  $g_{\mu\nu}(\eta) = a^2(\eta)\eta_{\mu\nu}$ ,  $\phi(\eta)$  results in a quadratic action for each physical component of the perturbation, which we denote by  $\psi$ ,

$$A = \frac{1}{2} \int d\eta d^3x a^{2m} e^{l\phi} (\psi'^2 - (\nabla\psi)^2 - M^2 a^2 \psi^2) \quad (2.4)$$

where  $'$  denotes  $\partial/\partial\eta$ ,  $M$  is the mass of the perturbed field and  $m$  and  $l$  depend on the type of field. We ignore complication associated with gauge invariance and work directly with physical components. For example, the background metric has 10 components of them 8 are non-physical, and each of the two physical components has an effective action of the form (2.4).

As already mentioned, we are interested in particles for which the mass  $M$  is either vanishing, or much smaller than  $M_s$ . In realistic models the mass may depend in a complicated way on the dilaton or other moduli. We will take  $M$  as an additional parameter and assume that it vanishes in the dilaton-driven phase and takes a constant value, much smaller than  $M_s$ , from the start of the RD era and on. The parameters  $l$  and  $m$  take the following values in some particular cases. For the dilaton or graviton  $m = 1$  and  $l = -1$ , for the model independent axion  $m = l = 1$ , for perturbative gauge fields such as photons,  $m = 0$  and  $l = -1$ , while for some non-perturbative gauge fields  $m = 0$  and  $l = 0$ , Ramond-Ramond axions and non-perturbative scalars have  $m = 1$  and  $l = 0$ . In general, fields with more tensor indices tend to have smaller (more negative)  $m$ 's. For example, the physical components of a 2-index tensor, whose field-strength is a 3-index tensor could have  $m = -1$ . We do not know at the moment a general rule for allowed values of  $m$  and  $l$ , but it is obvious that many combinations are possible.

The linearized equation of motion, satisfied by the field perturbation  $\psi$ , derived from the action (2.4) is the following,

$$(\Delta + M^2) \psi + \left( l + (m - 1) \left( 1 - 1/\sqrt{3} \right) \right) a^{-2} \psi' \phi' = 0 \quad (2.5)$$

where  $\Delta\psi$  is the covariant Laplacian and we have used the fact that for our particular case  $e^{\phi(\eta)} = a(\eta)^{3+\sqrt{3}}$ . In Fourier space  $\Delta\psi = a^{-2} [\psi_k'' + 2H\psi_k' + k^2\psi_k]$  and therefore the perturbation equation (2.5) is the following

$$\psi_k'' + \left[ 2H + \left( l + (m-1) \left( 1 - 1/\sqrt{3} \right) \right) \phi' \right] \psi_k' + (k^2 + M^2 a^2) \psi_k = 0. \quad (2.6)$$

Using the transformation  $\chi = S(\eta) \psi$  where  $S(\eta) = a(\eta)^m g(\eta)^l$  and  $g(\phi) = e^{\phi/2}$ , eq.(2.6) can be simplified,

$$\chi_k'' + \left( k^2 + M^2 a^2 - \frac{S''}{S} \right) \chi_k = 0. \quad (2.7)$$

The function  $S(\eta)$  was first introduced in [18].

### C. Approximate forms and solutions of the perturbation equation

Before trying to actually solve eq.(2.7) it is useful to look at approximate forms of the equation and their solutions in different physical situations.

A few special moments exist in the life of a perturbation, first,  $\eta \rightarrow -\infty$  when it is essentially a free field perturbation in flat space-time. Then, the time  $\eta_{ex}$  when the rate of expansion  $H$  becomes too fast for a given perturbation and it “exits” the horizon and its amplitude freezes (see below). The perturbation remains “outside the horizon” until the “reentry” time  $\eta_{re}$  when the expansion rate  $H$  is again slow enough and the perturbation defreezes. We are interested in perturbations that exit during the dilaton-driven phase  $\eta_{ex} < \eta_s$ , and reenter during RD,  $\eta_{re} > \eta_1$ . We do assume that all perturbations stay outside the horizon during the string phase, using in an essential way that the curvature during the string phase stays high.

In the range  $\eta < \eta_{ex}$ ,  $k^2 > S''/S \propto H^2$ , in the range  $\eta_{ex} < \eta < \eta_{re}$ ,  $k^2 + M^2 a^2 < S''/S \propto H^2$  and in the range  $\eta > \eta_{re}$ ,  $k^2 + M^2 a^2 > S''/S \propto H^2$ . Since we are interested in fields whose mass gets generated after the onset of RD and then have small masses, we observe that in the whole range  $-\infty < \eta < \eta_{re}$  the mass term in the perturbation equation may be ignored. In effect, in this range of time, we may use the approximate form

$$\chi_k'' + \left( k^2 - \frac{S''}{S} \right) \chi_k = 0. \quad (2.8)$$

The general solution of the perturbation equation is a linear combination of two modes, one which is approximately constant outside the horizon and one which is generically time dependent outside the horizon. We understand the appearance of a constant mode as the freezing of the perturbation amplitude, since local physics is no longer active on such scales. The existence of the time dependent mode can be most easily understood in terms of a constant mode of the conjugate momentum of  $\psi$ ,  $\Pi = S^2(\eta)\psi'$ . The amplitude of the conjugate momentum also freezes outside the horizon, since local physics is no longer active on such scales. This forces  $S^2(\eta)\psi'$  to be approximately constant leading to a “kinematical” time dependence of  $\psi$ . The perturbation for  $\Pi_k$  is simplified if rewritten for  $\chi_k^\pi = S^{-1}\Pi_k$

$$\chi_k^{\pi''} + \left( k^2 - \frac{(S^{-1})''}{S^{-1}} \right) \chi_k^\pi = 0 \quad (2.9)$$

and it's solutions are Bessel functions, as for  $\chi_k$ .

In the RD Universe (MD Universe)  $a = a_1 \eta / \eta_1$  ( $a = a_1 \eta_{eq} / \eta_1 (\eta / \eta_{eq})^2$ ), thus even if the  $S''/S$  is dominant at the beginning of the RD phase, then at later times  $M^2 a^2 + k^2$  becomes dominant and therefore in the range  $\eta > \eta_{re}$  the perturbation equation may be approximated by

$$\chi_k'' + (k^2 + M^2 a^2) \chi_k = 0. \quad (2.10)$$

The number density and energy density of the produced particles can be easily read off from the solutions of eq.(2.10) in a standard way [16].

We solve the perturbation equation using the following general method. We solve the early time equation with boundary conditions of normalized vacuum fluctuations. We assume that the constant mode of  $\psi$  and the constant mode of  $\Pi$  remain constant also while outside the horizon during the string phase, thus bridging the gap of unknown background evolution during the string phase. We then match the early time solutions to the late time solutions at  $\eta = \eta_1$  when both the early time and late time equation are expected to be good approximations. For massive particles, an additional matching step is performed at  $\eta = \eta_{re}$ . In all known cases our method reproduces results obtained by assuming specific background evolution during the string phase.

### III. SOLUTIONS OF THE PERTURBATION EQUATION

In this section we solve the perturbation equation, using the general procedure outlined in the previous section and obtain the necessary ingredients to assemble the spectra of produced particles.

#### A. Early time solutions

As explained previously, the approximate perturbation equation for early times is given by

$$\chi_k'' + \left(k^2 - \frac{S''}{S}\right) \chi_k = 0. \quad (3.1)$$

In the range  $\eta < \eta_s$ ,  $S(\eta) = a_s^m e^{l\phi_s/2} (\eta/\eta_s)^{1/2-n_s}$  where  $n_s = (\sqrt{3}(l+m) + 1 - m)/2$ , and in the range  $\eta > \eta_1$ ,  $S(\eta) = a_1^m e^{l\phi_1/2} (\eta/\eta_1)^{1/2-n_1}$  where  $n_1 = \frac{1}{2} - m$ . Therefore,  $S''/S = (n^2 - \frac{1}{4})\eta^{-2}$ , where  $n$  stands for either  $n_s$  or  $n_1$ , and the perturbation equation (3.1) takes the following simple form

$$\chi_k'' + \left(k^2 - (n^2 - \frac{1}{4})\eta^{-2}\right) \chi_k = 0. \quad (3.2)$$

The solutions of this equation are Bessel functions. For the special values  $n_s = \pm \frac{1}{2}$  (corresponding for example to  $m = 0, l = 0$ ), or  $n_1 = \pm \frac{1}{2}$  (corresponding to  $m = 0, 1$ ), the equation simplifies and the solutions are simple exponentials. It is nevertheless possible to continue to use the general setup and substitute these special values at the end.

Imposing that for  $k\eta \rightarrow -\infty$  the solution corresponds to vacuum fluctuations  $\frac{1}{\sqrt{2k}}e^{-ik\eta}$ , we obtain for  $\eta < \eta_s$

$$\chi_k(\eta) = \frac{\sqrt{\pi}}{2} \sqrt{\eta} J_n(k\eta) + \frac{\sqrt{\pi}}{2} \sqrt{\eta} Y_n(k\eta), \quad (3.3)$$

while for  $\eta > \eta_1$  we do not have any restrictions on the solution,

$$\chi_k = A\sqrt{\eta} J_n(k\eta) + B\sqrt{\eta} Y_n(k\eta). \quad (3.4)$$

For  $\eta < \eta_s$ , for perturbations outside the horizon,  $|k\eta| \lesssim 1$ ,  $\chi_k$  can be approximated by

$$\chi_k = \frac{\sqrt{\pi}}{2} \frac{(1 - i \cot(n_s \pi))}{\Gamma(1 + n_s)} \sqrt{\eta} \left(\frac{k\eta}{2}\right)^{n_s} \left[1 - \frac{(k\eta)^2}{2(2 + 2n_s)}\right] - i \frac{\sqrt{\pi}}{2} \frac{\csc(n_s \pi)}{\Gamma(1 - n_s)} \sqrt{\eta} \left(\frac{k\eta}{2}\right)^{-n_s} \left[1 - \frac{(k\eta)^2}{2(2 - 2n_s)}\right], \quad (3.5)$$

while for  $\eta > \eta_1$  and  $k\eta \lesssim 1$ ,  $\chi_k$  can be approximated by

$$\chi_k = A \frac{(1 - i \cot(n_1 \pi))}{\Gamma(1 + n_1)} \sqrt{\eta} \left(\frac{k\eta}{2}\right)^{n_1} \left[1 - \frac{(k\eta)^2}{2(2 + 2n_1)}\right] - i B \frac{\csc(n_1 \pi)}{\Gamma(1 - n_1)} \sqrt{\eta} \left(\frac{k\eta}{2}\right)^{-n_1} \left[1 - \frac{(k\eta)^2}{2(2 - 2n_1)}\right]. \quad (3.6)$$

Therefore, in the range  $\eta < \eta_s$  the solution is approximately given by

$$\begin{aligned} \psi_k^{in} &= \frac{2^{-1-n_s}}{\sqrt{\pi k_s}} \Gamma(-n_s) a_s^{-m} e^{-l\phi_s/2} (k_s k)^{n_s} \eta^{2n_s} \left[1 - \frac{(k\eta)^2}{2(2 + 2n_s)}\right] \\ &+ \frac{2^{n_s-1}}{\sqrt{\pi k_s}} \Gamma(n_s) a_s^{-m} e^{-l\phi_s/2} \left(\frac{k}{k_s}\right)^{-n_s} \left[1 - \frac{(k\eta)^2}{2(2 - 2n_s)}\right], \end{aligned} \quad (3.7)$$

where we have used the relations  $\Gamma(n)\Gamma(1-n) = \pi/\sin(n\pi)$  and  $\Gamma(-n)\Gamma(1+n) = \pi|1 - i \cot(\pi n)|$ , and defined  $k_s \equiv 1/|\eta_s|$ . In the range  $\eta > \eta_1$  we obtain

$$\begin{aligned} \psi_k^{out} &= A \frac{1}{2\pi \sqrt{k_1}} 2^{-n_1} \Gamma(-n_1) a_1^{-m} e^{-l\phi_1/2} (k_1 k)^{n_1} \eta^{2n_1} \left[1 - \frac{(k\eta)^2}{2(2 + 2n_1)}\right] \\ &+ B \frac{1}{\pi \sqrt{k_1}} 2^{n_1} \Gamma(n_1) a_1^{-m} e^{-l\phi_1/2} \left(\frac{k}{k_1}\right)^{-n_1} \left[1 - \frac{(k\eta)^2}{2(2 - 2n_1)}\right], \end{aligned} \quad (3.8)$$

where we have used the relations  $\Gamma(n)\Gamma(1-n) = \pi/\sin(n\pi)$  and  $\Gamma(-n)\Gamma(1+n) = \pi|1-i\cot(\pi n)|$  and defined  $k_1 \equiv 1/\eta_1$ . Note that we have kept, for later use, also the next to leading terms. The corrections are of order  $(k\eta)^2$ , which is a very small quantity, showing that the approximation we use is quite good.

The perturbation equation for  $\chi^\pi$  is the following

$$\chi_k^{\pi''} + \left(k^2 - \frac{(S^{-1})''}{S^{-1}}\right) \chi_k^\pi = 0, \quad (3.9)$$

and it's solutions are Bessel functions, as for  $\chi_k$ . We normalize the solution at early times to vacuum fluctuations,  $\frac{\sqrt{k}}{\sqrt{2}}e^{-ik\eta}$  (note the  $\sqrt{k}$  instead of  $1/\sqrt{k}$  in the normalization of the momentum). Substituting the explicit expressions for  $S(\eta) = a^m(\eta)e^{l\phi(\eta)/2}$ , where  $a(\eta)$  and  $\phi(\eta)$  are given in eq.(2.1), eq.(3.9) takes a form similar to eq.(3.2), except that instead of  $n_s$  appears  $n_s^\pi$  and instead of  $n_1$  appears  $n_1^\pi$ , where  $n_s^\pi = n_s - 1 = (\sqrt{3}(l+m) - 1 - m)/2$  and  $n_1^\pi = n_1 - 1 = -\frac{1}{2} - m$ . The solution of eq.(3.9) for  $\eta < \eta_s$  is the following

$$\chi_k^\pi = \frac{\sqrt{\pi}}{2} \sqrt{\eta} J_{n_s^\pi}(k\eta) + \frac{\sqrt{\pi}}{2} \sqrt{\eta} Y_{n_s^\pi}(k\eta), \quad (3.10)$$

and for  $\eta > \eta_1$

$$\chi_k^\pi = A\sqrt{\eta} J_{n_1^\pi}(k\eta) + B\sqrt{\eta} Y_{n_1^\pi}(k\eta). \quad (3.11)$$

For the special values  $n_s^\pi = \pm\frac{1}{2}$  and  $n_1^\pi = \pm\frac{1}{2}$  the solution reduces to a sum of exponentials. It is nevertheless possible to use the general formulae for these special values as well.

For  $\eta < \eta_s$ , for perturbations outside the horizon,  $|k\eta| \lesssim 1$ ,  $\chi^\pi$  can be approximated by

$$\chi_k^\pi = \frac{\Gamma(-n_s^\pi)}{2\sqrt{\pi}} k \sqrt{\eta} \left(\frac{k\eta}{2}\right)^{n_s^\pi} \left[1 - \frac{(k\eta)^2}{2(2+2n_s^\pi)}\right] - i \frac{\Gamma(n_s^\pi)}{2\sqrt{\pi}} k \sqrt{\eta} \left(\frac{k\eta}{2}\right)^{-n_s^\pi} \left[1 - \frac{(k\eta)^2}{2(2-2n_s^\pi)}\right], \quad (3.12)$$

while for  $\eta > \eta_1$ ,  $k\eta \lesssim 1$ ,  $\chi^\pi$  can be approximated by

$$\chi_k^\pi = A_\pi \frac{\Gamma(-n_1^\pi)}{\pi} \sqrt{\eta} \left(\frac{k\eta}{2}\right)^{n_1^\pi} \left[1 - \frac{(k\eta)^2}{2(2+2n_1^\pi)}\right] - i B_\pi \frac{\Gamma(n_1^\pi)}{\pi} \sqrt{\eta} \left(\frac{k\eta}{2}\right)^{-n_1^\pi} \left[1 - \frac{(k\eta)^2}{2(2-2n_1^\pi)}\right]. \quad (3.13)$$

Since for  $\eta < \eta_s$ ,  $\Pi = S\chi^\pi = a_s^m e^{l\phi_s/2} (\eta/\eta_s)^{1/2-n_s} \chi^\pi$ ,

$$\begin{aligned} \Pi_k^{in} &= \frac{2^{-1-n_s^\pi}}{\sqrt{\pi}} \Gamma(-n_s^\pi) a_s^m e^{l\phi_s/2} (k_s)^{1/2} \left(\frac{k}{k_s}\right)^{n_s} \left[1 - \frac{(k\eta)^2}{2(2+2n_s^\pi)}\right] \\ &\quad - i \frac{2^{n_s^\pi-1}}{\sqrt{\pi}} \Gamma(n_s^\pi) a_s^m e^{l\phi_s/2} k (k_s)^{-1/2} (k_s k)^{-n_s+1} \eta^{-2n_s^\pi} \left[1 - \frac{(k\eta)^2}{2(2-2n_s^\pi)}\right], \end{aligned} \quad (3.14)$$

and since for  $\eta > \eta_1$ ,  $\Pi = S\chi^\pi = a_1^m e^{l\phi_1/2} (\eta/\eta_1)^{1/2-n_1} \chi^\pi$ ,

$$\begin{aligned} \Pi_k^{out} &= \frac{2^{-n_1^\pi}}{\pi} A_\pi \Gamma(-n_1^\pi) a_1^m e^{l\phi_1/2} k^{-1} k_1^{1/2} \left(\frac{k}{k_1}\right)^{n_1} \left[1 - \frac{(k\eta)^2}{2(2+2n_1^\pi)}\right] \\ &\quad - i \frac{2^{n_1^\pi}}{\pi} B_\pi \Gamma(n_1^\pi) a_1^m e^{l\phi_1/2} k_1^{-1/2} (k_1 k)^{-n_1+1} \eta^{-2n_1^\pi} \left[1 - \frac{(k\eta)^2}{2(2-2n_1^\pi)}\right]. \end{aligned} \quad (3.15)$$

### 1. Bridging the gap: The constant mode of $\psi$

We will compute the contribution to the *out* perturbation from the constant mode of  $\psi^{in}$ . From the second term of eq.(3.7) we obtain the the constant mode of  $\psi^{in}$

$$\psi_k^{in} = \frac{1}{\sqrt{\pi k_s}} 2^{n_s-1} \Gamma(n_s) a_s^{-m} e^{-l\phi_s/2} \left(\frac{k}{k_s}\right)^{-n_s} \quad (3.16)$$

For  $\eta > \eta_1$ ,  $\psi^{out}$  is given in eq.(3.8). We assume that the constant mode of  $\psi$  is indeed constant during the string phase (and, in general, outside the horizon), as explained in the previous section, and use this to connect the two solutions by matching them and their first derivatives at  $\eta = \eta_1$ . This matching procedure is expected to be quite accurate because we match the solutions at a point in time at which both solutions are well outside the horizon. We present the matching equations once explicitly, to emphasize the importance of keeping the next to leading terms in eqs.(3.7) and (3.8).

$$\begin{aligned} & \frac{1}{2\sqrt{\pi k_s}} 2^{n_s} \Gamma(n_s) a_s^{-m} e^{-l\phi_s/2} \left(\frac{k}{k_s}\right)^{-n_s} = \\ & B \frac{2^{n_1}}{\pi\sqrt{k_1}} \Gamma(n_1) a_1^{-m} e^{-l\phi_1/2} \left(\frac{k}{k_1}\right)^{-n_1} + A \frac{2^{-n_1}}{2\pi\sqrt{k_1}} \Gamma(-n_1) a_1^{-m} e^{-l\phi_1/2} (k_1 k)^{n_1} \eta_1^{2n_1}, \\ 0 = & -B \frac{2}{\eta_1} \frac{(k\eta_1)^2}{2(2-2n_1)} \frac{2^{n_1}}{\pi\sqrt{k_1}} \Gamma(n_1) a_1^{-m} e^{-l\phi_1/2} \left(\frac{k}{k_1}\right)^{-n_1} + \\ & A \frac{2^{-n_1}}{2\pi\sqrt{k_1}} \Gamma(-n_1) a_1^{-m} e^{-l\phi_1/2} (k_1 k)^{n_1} \eta_1^{2n_1} \frac{2n_1}{\eta_1}. \end{aligned} \quad (3.17)$$

The solution of eq.(3.17) is

$$A = \frac{2^{n_s+n_1-2}\sqrt{\pi}}{n_1(1-n_1)} \frac{\Gamma(n_s)}{\Gamma(-n_1)} z_S^{-3/2+m+n_1} \left(\frac{g_S}{g_1}\right)^{-l} \left(\frac{k}{k_s}\right)^{2-n_1-n_s} \quad (3.18)$$

$$B = 2^{n_s-n_1-1} \sqrt{\pi} \frac{\Gamma(n_s)}{\Gamma(n_1)} z_S^{1/2+m-n_1} \left(\frac{g_S}{g_1}\right)^{-l} \left(\frac{k}{k_s}\right)^{n_1-n_s}. \quad (3.19)$$

To obtain eqs.(3.18),(3.19) we have used the definitions  $z_S = a_1/a_S$ ,  $g_1 = e^{\phi_1/2}$ ,  $g_S = e^{\phi_S/2}$ . Note that had we not kept the next to leading terms in eq.(3.7) we would have obtained  $A = 0$ .

To summarize, plugging in the resulting coefficients from eqs.(3.18) and (3.19) into eq.(3.8) we obtain

$$\psi_k^{out} = \frac{1}{\sqrt{\pi k_s}} 2^{n_s-1} \Gamma(n_s) a_s^{-m} g_S^{-l} z_S^{2m} \left(\frac{k}{k_s}\right)^{-n_s} + \frac{1}{\sqrt{\pi k_s}} \frac{2^{n_s-2} \Gamma(n_s)}{n_1(1-n_1)} a_s^{-m} g_S^{-l} z_S^{n_1-1} \left(\frac{k}{k_s}\right)^{2-n_s} (k_s \eta)^{2n_1} \quad (3.20)$$

The *out* solution (3.20) may be put, using the relation  $k_S z_S = k_1$ , in the form  $\psi_k^{out} = C_1 \left[ 1 + C_2 \left(\frac{k}{k_1}\right)^2 (k_1 \eta)^{2n_1} \right]$ , where  $C_2$  is a numerical coefficient of order 1. At  $\eta = \eta_1$ , the ratio of the two terms is approximately  $(k\eta_1)^2 \ll 1$ , because  $k_1 \eta_1 = 1$ . However, at later times, using eq.(3.4), the general solution is given by  $\psi_k^{out} \simeq A S^{-1}(\eta) \sqrt{\eta} J_{n_1}(k\eta) + B S^{-1}(\eta) \sqrt{\eta} Y_{n_1}(k\eta)$ . A good time to look at is near  $\eta_{re}$ . At that time the magnitude of both Bessel functions is generically the same, and to decide on the leading contribution we look at the ratio of the coefficients  $A/B \simeq \left(\frac{k}{k_1}\right)^{2-2n_1}$ . Therefore the leading contribution at much later times depends on  $n_1$ . For  $n_1 > 1$ ,  $A \gg B$  and the leading contribution will be from the time dependent term in eq.(3.20), and for  $n_1 < 1$ ,  $B \gg A$  and the leading contribution will be from the constant term in eq.(3.20). In conclusion, for later times,

$$\psi_k^{out} \simeq A S^{-1}(\eta) \sqrt{\eta} J_{n_1}(k\eta) \quad n_1 > 1 \quad (3.21)$$

$$\psi_k^{out} \simeq B S^{-1}(\eta) \sqrt{\eta} Y_{n_1}(k\eta) \quad n_1 < 1, \quad (3.22)$$

where  $A$  and  $B$  are given by eq.(3.18) and eq.(3.19).

## 2. Bridging the gap: The time dependent mode of $\psi$

We will compute the contribution to the *out* perturbation from the time dependent mode of  $\psi^{in}$ . The time dependent mode of  $\psi$  can be extracted from the first term in eq.(3.7),

$$\psi_k^{in} = \frac{1}{2\sqrt{\pi k_s}} 2^{-n_s} \Gamma(-n_s) a_s^{-m} e^{-l\phi_s/2} (k_s k)^{n_s} \eta_1^{2n_s} \quad (3.23)$$

Since for  $\eta < \eta_s$ ,  $S^2(\eta) = a_s^{2m} e^{l\phi_s} (\eta/\eta_s)^{1-2n_s}$ , we obtain



$$\psi_k^{in'} S^2 = \frac{n_s \sqrt{k_s}}{\sqrt{\pi}} 2^{-n_s} \Gamma(-n_s) a_s^m e^{l\phi_s/2} \left(\frac{k}{k_s}\right)^{n_s} = \text{const.} \quad (3.24)$$

We can see explicitly that the time dependent mode of  $\psi$  corresponds to the constant mode of the canonically conjugate momentum  $\Pi$ .

We compute the contribution from the time dependent mode of  $\psi$  using the constant mode of  $\Pi$ , assuming that it is constant throughout the string phase (and, in general, outside the horizon), as explained in the previous section, and use this to connect the two solutions by matching them and their first derivatives at  $\eta = \eta_1$ . This matching procedure is expected to be quite accurate because we match the solutions at a point in time at which both solutions are well outside the horizon.

The solutions are given in eq.(3.14) and (3.15) and their matching yields

$$A_\pi = \sqrt{\pi} 2^{n_1 - n_s - 1} \frac{\Gamma(1 - n_s)}{\Gamma(1 - n_1)} z_S^{-1/2 - m + n_1} \left(\frac{g_S}{g_1}\right)^l k \left(\frac{k}{k_s}\right)^{-n_1 + n_s} \quad (3.25)$$

$$B_\pi = \sqrt{\pi} \frac{2^{3n_1 - n_s - 2}}{n_1(1 - 2n_1)} \frac{\Gamma(1 - n_s)}{\Gamma(n_1 - 1)} z_S^{-1/2 - m - n_1} \left(\frac{g_S}{g_1}\right)^l k \left(\frac{k}{k_s}\right)^{n_1 + n_s}, \quad (3.26)$$

where we have used the definitions of  $n_s^\pi$  and  $n_1^\pi$ . Substituting into eq.(3.15) we obtain

$$\Pi_k^{out} = \frac{2^{1-n_1}}{\pi} A_\pi \Gamma(1 - n_1) a_1^m e^{l\phi_1/2} k^{-1} k_1^{1/2} \left(\frac{k}{k_1}\right)^{n_1} - i \frac{2^{n_1-1}}{\pi} B_\pi \Gamma(n_1 - 1) a_1^m e^{l\phi_1/2} k_1^{-1/2} (k_1 k)^{-n_1+1} \eta^{2-2n_1} \quad (3.27)$$

with  $A_\pi$  and  $B_\pi$  given in eq.(3.25) and (3.26).

As for the case of the constant mode of  $\psi$ , we may compare the relative strength of the two terms. At  $\eta = \eta_1$ , the relative strength of the two terms is approximately  $(k\eta_1)^2 \ll 1$ , because  $k_1\eta_1 = 1$ . However, at later times, using eq.(3.11), the general solution is given by  $\Pi_k^{out} \simeq A_\pi S(\eta) \sqrt{\eta} J_{n_1^\pi}(k\eta) + B_\pi S(\eta) \sqrt{\eta} Y_{n_1^\pi}(k\eta)$ . A good time to look at is near  $\eta_{re}$ . At that time the magnitude of both Bessel functions is generically the same and to decide on the leading contribution we look at the ratio of the coefficients  $A_\pi/B_\pi \simeq \left(\frac{k}{k_1}\right)^{-2n_1}$ . Therefore the leading contribution at much later times depends on  $n_1$ . For  $n_1 > 0$ ,  $A_\pi \gg B_\pi$  and the leading contribution will be from the time dependent term in eq.(3.27), and for  $n_1 < 0$ ,  $B_\pi \gg A_\pi$  and the leading contribution will be from the constant term in eq.(3.27). In conclusion, for later times,

$$\begin{aligned} \Pi_k^{out} &\simeq S(\eta) A_\pi \sqrt{\eta} J_{n_1^\pi}(k\eta) \text{ for } n_1 > 0 \\ \Pi_k^{out} &\simeq S(\eta) B_\pi \sqrt{\eta} Y_{n_1^\pi}(k\eta) \text{ for } n_1 < 0 \end{aligned} \quad (3.28)$$

where  $A_\pi$  and  $B_\pi$  are given in eqs.(3.25), (3.26).

## B. Late time solutions and spectrum: The massless approximation

The perturbation equation in the range  $\eta > \eta_{re}$  is  $\chi_k'' + \left(k^2 + M^2 a^2 - \frac{S''}{S}\right) \chi_k = 0$ . Because we cannot solve this equation in a closed form for the general case, we use an approximate form of the equation to obtain approximate solutions. We distinguish between two cases,  $k > Ma_{re}$  - the ‘‘massless approximation’’;  $k < Ma_{re}$  - the ‘‘massive approximation’’, where  $a_{re} = a(\eta_{re})$ .

In the massless approximation the perturbation equation is  $\chi_k'' + \left(k^2 - \frac{S''}{S}\right) \chi_k = 0$ , which is the same equation we solved already, so there's no need for any additional steps. All we need to do is to take the late time solutions and interpret them. In a standard way, we compute from the solution the number of produced particles at a given wave number  $k$ ,  $N_k$ , and the ratio of the energy density per octave of the produced particles to the critical energy density  $\frac{d\Omega}{d \ln \omega}$ . For massless particles  $d\Omega/(d \ln \omega) = w^4 N_\omega / (M_p H_0)^2$ , where  $H_0$  is today's Hubble parameter,  $M_p$  is the Planck mass and  $\omega = k/a(\eta)$  [24].

We can further simplify the denominator  $(M_p H_0)^2$ . For that we switch momentarily to using cosmic time. We may reexpress the relation  $H_0 = H_{eq} \frac{t_{eq}}{t_0}$ , as  $H_0 = H_1 \frac{t_1}{t_{eq}} \frac{t_{eq}}{t_0}$ , where  $H_1$  is the Hubble parameter at  $t_1$ , the onset of RD. In MD  $a(t) = a_{eq} \left(\frac{t}{t_{eq}}\right)^{2/3}$  and therefore  $\frac{t_{eq}}{t_0} = \left(\frac{a_0}{a_{eq}}\right)^{-3/2}$ . In RD  $a_{eq} = a_1 \left(\frac{t_{eq}}{t_1}\right)^{1/2}$ , then  $\frac{t_1}{t_{eq}} = \left(\frac{a_{eq}}{a_1}\right)^{-2}$ . Combining these relations, we obtain  $H_0 = H_1 \left(\frac{a_{eq}}{a_1}\right)^{-2} \left(\frac{a_0}{a_{eq}}\right)^{-3/2}$ . Since we assumed that  $H_1 = M_s$  then  $H_0 = M_s \left(\frac{a_1}{a_0}\right)^2 \left(\frac{a_0}{a_{eq}}\right)^{1/2}$ .

Now,  $M_p = M_s g_1^{-1}$ ,  $M_s = \omega_1 a_0 / a_1$  and  $\omega_1 = \omega_s z_S$ , and therefore  $M_s \frac{a_1}{a_0} = \omega_s z_S$ . Combining these relations we arrive at our final destination

$$\begin{aligned} M_p H_0 &= M_s^2 \left( \frac{a_1}{a_0} \right)^2 z_{eq}^{1/2} g_1^{-1} \\ &= \omega_s^2 z_S^2 z_{eq}^{1/2} g_1^{-1}, \end{aligned} \quad (3.29)$$

where  $z_{eq} = a_0 / a_{eq}$ .

Substituting eq.(3.29) into the expression for the energy density we obtain the following expression

$$d\Omega / (d \ln \omega) = z_{eq}^{-1} z_S^{-4} g_1^2 \left( \frac{\omega}{\omega_s} \right)^4 N_\omega. \quad (3.30)$$

The factor  $z_{eq}^{-1}$ , the redshift from the time of matter-radiation equality and today, appears because the energy density of massless particles redshifts faster than that of matter.

We may also discuss spectra of massive particles in the massless approximation. Although this may sound a little strange, such an approximation does make sense, because it is made at  $\eta_{re}$ , when the effects of the mass can be negligible. The only difference between massive particles and massless particles in the massless approximation is in their energy density. The solutions near  $\eta_{re}$  are approximately the same, and therefore the number of produced particles is the same as in the massless case. However, at much later times, the contribution from the mass may dominate the energy density. We can then compute the ratio of the energy density per octave of the produced particles to the critical energy density  $\frac{d\Omega}{d \ln \omega}$ . For massive particles  $d\Omega / (d \ln \omega) = \sqrt{M^2 + \omega^2} \omega^3 N_\omega / (M_p H_0)^2$ , where  $H_0$  is today's Hubble parameter,  $M_p$  is the Planck mass and  $\omega = k/a(\eta)$ . Using eq.(3.29) we may show that

$$d\Omega / (d \ln \omega) = z_{eq}^{-1} z_S^{-4} g_1^2 \frac{\sqrt{M^2 + \omega^2}}{\omega_s} \left( \frac{\omega}{\omega_s} \right)^3 N_\omega. \quad (3.31)$$

An important difference compared with the massless spectrum is that one power of  $\omega/\omega_s$  is replaced by  $\frac{\sqrt{M^2 + \omega^2}}{\omega_s}$  (for  $\omega < M$ ). Because the energy density of massive particles redshifts slower than that of radiation, two enhancement factors will appear compared with the case of massless particles. One enhancement factor accounts for the period between reentry and matter-radiation equality, in which the relative energy density of massive particles grows, while that of massless particles remains constant. Another enhancement factor appears because the relative energy density of massive particles stays constant during the matter dominated era, while that of massless particles decreases. It is harder to see this factor in the equations because  $z_{eq}$  also appears in the expressions for massive particles. However, it's appearance is artificial. The combination  $M/(z_{eq} \omega_s) = M/\omega_s(\eta_{eq})$  actually does not depend on the redshift from matter-radiation equality epoch until today.

### 1. Massless spectrum: contribution of the constant mode of $\psi$

The approximate late time solutions of the perturbation equation are given in eqs.(3.21), (3.22). Using the asymptotic expansion of Bessel functions, the solutions (either during RD or MD) approach, for  $k\eta > 1$

$$\psi_k = S^{-1} \frac{1}{\sqrt{2k}} (\alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}) \quad (3.32)$$

where

$$|\alpha_k| = |\beta_k| = 2^{n_s - n_1} \left| \frac{\Gamma(n_s)}{\Gamma(n_1)} \right| z_S^{1/2 + m - n_1} \left( \frac{g_S}{g_1} \right)^{-l} \left( \frac{k}{k_s} \right)^{n_1 - n_s} \quad \text{for } n_1 < 1 \quad (3.33)$$

and

$$|\alpha_k| = |\beta_k| = \frac{2^{n_s + n_1 - 1}}{n_1(1 - n_1)} \left| \frac{\Gamma(n_s)}{\Gamma(-n_1)} \right| z_S^{-3/2 + m + n_1} \left( \frac{g_S}{g_1} \right)^{-l} \left( \frac{k}{k_s} \right)^{2 - n_1 - n_s} \quad \text{for } n_1 > 1 \quad (3.34)$$

The number of massless particles  $N_k$ , can be expressed in the standard form  $N_k = |\beta_k|^2$ . Using eq.(3.30),  $n_1 = \frac{1}{2} - m$ , and using  $d\Omega_\psi$  to denote the contribution from the constant mode of  $\psi$  we arrive at the final expressions

$$\left[ \frac{d\Omega_\psi}{d\ln\omega} \right]_{n_1 < 1} = 2^{2n_s+2m-1} \frac{\Gamma^2(n_s)}{\Gamma^2(\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{4m-4} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{5-2m-2n_s} \quad (3.35)$$

$$\left[ \frac{d\Omega_\psi}{d\ln\omega} \right]_{n_1 > 1} = \frac{2^{2n_s-2m-1}}{(\frac{1}{4}-m^2)^2} \frac{\Gamma^2(n_s)}{\Gamma^2(m-\frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-6} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{7+2m-2n_s}. \quad (3.36)$$

## 2. Massless spectrum: contribution of the time dependent mode of $\psi$

As usual, we use the constant mode of  $\Pi$  to evaluate the contribution of the time dependent mode of  $\psi$ . The approximate late time solutions of the perturbation equation are given in eq.(3.28). Using the asymptotic expansion of Bessel functions, the solutions (either during RD or MD) approach, for  $k\eta > 1$

$$\Pi_k = S \frac{\sqrt{k}}{\sqrt{2}} (\alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}) \quad (3.37)$$

where

$$|\alpha_k| = |\beta_k| = 2^{n_1-n_s} \left| \frac{\Gamma(-n_s^\pi)}{\Gamma(-n_1^\pi)} \right| z_S^{-1/2-m+n_1} \left( \frac{g_S}{g_1} \right)^l \left( \frac{k}{k_s} \right)^{-n_1+n_s} \quad \text{for } n_1 > 0 \quad (3.38)$$

and

$$|\alpha_k| = |\beta_k| = \frac{2^{3n_1-n_s-1}}{|n_1(1-2n_1)|} \left| \frac{\Gamma(-n_s^\pi)}{\Gamma(n_1^\pi)} \right| z_S^{-1/2-m-n_1} \left( \frac{g_S}{g_1} \right)^l \left( \frac{k}{k_s} \right)^{n_1+n_s} \quad \text{for } n_1 < 0. \quad (3.39)$$

The number of massless particles  $N_k$  of a given wavenumber  $k$ , can be expressed as  $N_k = |\beta_k|^2$ . Note that we are calculating the number of particles directly from the solution for  $\Pi_k$ . We could have also used the relation  $\Pi = S^2 \psi'$ , use the standard definition of  $N_k$  from the coefficients in the solution for  $\psi$  and obtain the same result.

Using eq.(3.30),  $n_1 = \frac{1}{2} - m$ , and using  $d\Omega_\pi$  to denote the contribution from the time dependent mode of  $\psi$  (the constant mode of  $\Pi$ ) we arrive at the final expressions

$$\left[ \frac{d\Omega_\pi}{d\ln\omega} \right]_{n_1 > 0} = 2^{1-2m-2n_s} \frac{\Gamma^2(1-n_s)}{\Gamma^2(m+\frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-4m-4} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{3+2m+2n_s} \quad (3.40)$$

$$\left[ \frac{d\Omega_\pi}{d\ln\omega} \right]_{n_1 < 0} = \frac{2^{-6m-2n_s-5}}{m^2(m-\frac{1}{2})^2} \frac{\Gamma^2(1-n_s)}{\Gamma^2(-\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{-6} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{5-2m+2n_s}. \quad (3.41)$$

## 3. Massive spectrum: contribution of the constant mode of $\psi$

The final result for the contribution of the constant mode of  $\psi$  to the spectrum of massive particles (recall that we are using the massless approximation, defined at the beginning of this section) maybe read off from the final result for massless particles, eqs.(3.35) and (3.36), by making the substitution of one power of  $\omega/\omega_s$  by one power of  $\frac{\sqrt{M^2+\omega^2}}{\omega_s}$ ,

$$\left[ \frac{d\Omega_\psi}{d\ln\omega} \right]_{n_1 < 1} = 2^{2n_s+2m-1} \frac{\Gamma^2(n_s)}{\Gamma^2(\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{4m-4} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{4-2m-2n_s} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \quad (3.42)$$

$$\left[ \frac{d\Omega_\psi}{d\ln\omega} \right]_{n_1 > 1} = \frac{2^{2n_s-2m-1}}{(\frac{1}{4}-m^2)^2} \frac{\Gamma^2(n_s)}{\Gamma^2(m-\frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-6} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{6+2m-2n_s} \frac{\sqrt{M^2+\omega^2}}{\omega_s}. \quad (3.43)$$

#### 4. Massive spectrum: contribution of the time dependent mode of $\psi$

The final result for the contribution of the constant mode of  $\psi$  to the spectrum of massive particles (recall that we are using the massless approximation, defined at the beginning of this section) maybe read off from the final result for massless particles, eqs.(3.40) and (3.41), by making the substitution of one power of  $\omega/\omega_s$  by one power of  $\frac{\sqrt{M^2+\omega^2}}{\omega_s}$ ,

$$\left[ \frac{d\Omega_\pi}{d\ln\omega} \right]_{n_1>0} = 2^{1-2m-2n_s} \frac{\Gamma^2(1-n_s)}{\Gamma^2(m+\frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-4m-4} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{2+2m+2n_s} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \quad (3.44)$$

$$\left[ \frac{d\Omega_\pi}{d\ln\omega} \right]_{n_1<0} = \frac{2^{-6m-2n_s-5}}{m^2(m-\frac{1}{2})^2} \frac{\Gamma^2(1-n_s)}{\Gamma^2(-\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{-6} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{4-2m+2n_s} \frac{\sqrt{M^2+\omega^2}}{\omega_s}. \quad (3.45)$$

### C. Late time solutions and spectrum: The massive approximation

If the produced particles are massive, and at  $\eta_{re}$ ,  $Ma_{re} > k$ , we are unable to solve the late time equation exactly, and therefore need to resort to another matching procedure. We choose to perform the matching at  $\eta = \eta_{re}$ , where  $k^2 + M^2 a_{re}^2 \sim H^2$ . We match a solution from outside the horizon to a massive solution inside. This is not an ideal match, but both are expected to be fairly accurate at  $\eta_{re}$ . Note that the solutions outside the horizon were obtained using a “massless” equation, however outside the horizon the dominant term is  $S''/S$  anyhow, and therefore no large inaccuracies should be induced. We then need to know whether  $\eta_{re}$  occurs during RD or MD. If  $Ma_{eq} > H(\eta_{eq})$  then the reentry time of all perturbation will be at earlier times. Therefore, for  $M > 10^{-27}eV$ , the reentry time is during RD. Since we do not expect any particles to have masses smaller than  $10^{-27}eV$ , we continue under the assumption that  $\eta_{re} < \eta_{eq}$ . We have performed the calculations for the case  $\eta_{re} > \eta_{eq}$ , however, we will not present them in this paper.

In a standard way, we compute from the solution the number of produced particles at a given wave number  $k$ ,  $N_k$ . We then compute the ratio of the energy density per octave of the produced particles to the critical energy density  $\frac{d\Omega}{d\ln\omega}$ . As already shown, for massive particles  $d\Omega/(d\ln\omega)$  is given by eq.(3.31).

The equation of motion in the massive approximation is

$$\chi_k'' + M^2 a^2 \chi_k = 0 \quad (3.46)$$

and it's solutions in the WKB approximation are given by

$$\chi_k^\pm = \frac{1}{\sqrt{2Ma(\eta)}} \exp\left(\pm i \int^\eta Ma(\eta') d\eta'\right), \quad (3.47)$$

where for the RD epoch ( $\eta < \eta_{eq}$ )

$$\chi_k^\pm = \frac{1}{\sqrt{2Ma_1 k_1 \eta}} \exp\left(\pm i \frac{1}{2} Ma_1 k_1 \eta^2\right), \quad (3.48)$$

Since  $\psi = S^{-1}\chi$  the general solution for the massive perturbation for RD is the following,

$$\psi_k^{massive} = \frac{1}{\sqrt{2Ma_1 k_1 \eta}} S^{-1}(\eta) \left[ \alpha_k \exp\left(-i \frac{1}{2} Ma_1 k_1 \eta^2\right) + \beta_k \exp\left(i \frac{1}{2} Ma_1 k_1 \eta^2\right) \right]. \quad (3.49)$$

We will also need the general solution for the conjugate momentum, which can either be derived from the equation for  $\chi^\pi$  or from the relation  $\Pi = S^2 \psi'$ , both yielding the approximate solution

$$\Pi_k^{massive} = \sqrt{2Ma_1 k_1 \eta} S(\eta) \left[ \alpha_k^\pi \exp\left(-i \frac{1}{2} Ma_1 k_1 \eta^2\right) + \beta_k^\pi \exp\left(i \frac{1}{2} Ma_1 k_1 \eta^2\right) \right]. \quad (3.50)$$

Instead of using only  $\psi$  at late times, we will again use  $\Pi$ , making sure that we do not double count the number of produced particles.

1. Massive spectrum: The contribution of the constant mode of  $\psi$

As in the massless case, we assume that  $H_1^2 > k^2 + M^2 a_1^2$ , for all  $k$ , therefore solutions for  $\eta > \eta_1$  are given in eqs.(3.21), (3.22). We reproduce here their approximate forms,

$$\psi_k^{out} = S^{-1}(\eta) \frac{\sqrt{\pi} 2^{n_s+n_1-2}}{n_1(1-n_1)} \frac{\Gamma(n_s)}{\Gamma(-n_1)} z_S^{-1} \left(\frac{g_S}{g_1}\right)^{-l} \left(\frac{k}{k_s}\right)^{2-n_1-n_s} \sqrt{\eta} J_{n_1}(k\eta) \text{ for } n_1 > 1 \quad (3.51)$$

$$\psi_k^{out} = S^{-1}(\eta) \sqrt{\pi} 2^{n_s-n_1-1} \frac{\Gamma(n_s)}{\Gamma(n_1)} z_S^{2m} \left(\frac{g_S}{g_1}\right)^{-l} \left(\frac{k}{k_s}\right)^{n_1-n_s} \sqrt{\eta} J_{-n_1}(k\eta) \text{ for } n_1 < 1 \quad (3.52)$$

In the case of the massive approximation these solutions are correct only for  $H^2 > k^2 + M^2 a^2$ . Therefore for  $k\eta < 1$  (and  $M < k/a$ ) the leading terms are

$$\psi_{n_1>1} = \frac{1}{\sqrt{\pi} k_s} \frac{2^{n_s-2} \Gamma(n_s)}{n_1(1-n_1)} a_s^{-m} z_S^{2(n_1-1)} g_S^{-l} \left(\frac{k}{k_s}\right)^{2-n_s} (k_s \eta)^{2n_1} \quad (3.53)$$

for the time dependent mode, and

$$\psi_{n_1<1} = \frac{1}{\sqrt{\pi} k_s} 2^{n_s-1} \Gamma(n_s) a_s^{-m} g_S^{-l} \left(\frac{k}{k_s}\right)^{-n_s} \quad (3.54)$$

for the constant mode.

(a) *The case  $n_1 > 1$ .*

In the case  $n_1 > 1$ , the approximate solution for  $\eta \lesssim \eta_{re}$  is given in eq.(3.53), and for  $\eta \gtrsim \eta_{re}$  we obtained the approximate solution in eq.(3.49). We connect the solutions at  $\eta = \eta_{re}$  by matching them and their first derivatives,  $\psi_k^{out}(\eta_{re}) = \psi_k^{massive}(\eta_{re})$  and  $\psi_k^{out}(\eta_{re})' = \psi_k^{massive}(\eta_{re})'$ . Using the relations  $H(\eta_{re}) = Ma(\eta_{re})$  and  $\eta_{re}^{-1} \equiv k_{re} = (Ma_1 k_1)^{1/2}$ , the results of the matching give

$$|\beta_k| = \frac{|3m+i|}{\sqrt{\pi}} \frac{2^{n_s-3} |\Gamma(n_s)|}{|n_1(1-n_1)|} z_S^{-1} \left(\frac{g_S}{g_1}\right)^{-l} \left(\frac{k}{k_s}\right)^{2-n_s} \left(\frac{k_{re}}{k_s}\right)^{-n_1}. \quad (3.55)$$

Since  $k_{re}/k_s = (Ma_1/k_1)^{1/2} z_S$ , and since we assume as in [21]  $k_1/a_1 = M_s$ , we obtain

$$|\beta_k| = \frac{|3m+i|}{\sqrt{\pi}} \frac{2^{n_s-3} |\Gamma(n_s)|}{|n_1(1-n_1)|} z_S^{m-3/2} \left(\frac{g_S}{g_1}\right)^{-l} \left(\frac{\omega}{\omega_s}\right)^{2-n_s} \left(\frac{M}{M_s}\right)^{-n_1/2}. \quad (3.56)$$

Using eq.(3.31), and the definition of  $n_1$ , we obtain the final result

$$\left[ \frac{d\Omega_\psi}{d\ln\omega} \right]_{n_1>1} = \frac{|3m+i|^2}{\pi} \frac{2^{2n_s-6}}{(\frac{1}{4}-m^2)^2} \Gamma^2(n_s) z_{eq}^{-1} g_1^2 z_S^{2m-7} \left(\frac{g_S}{g_1}\right)^{-2l} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^{7-2n_s} \left(\frac{M}{M_s}\right)^{m-1/2}. \quad (3.57)$$

(b) *The case  $n_1 < 1$ .*

In the case  $n_1 < 1$  we obtained the approximate form of the solution for  $\eta \lesssim \eta_{re}$  in eq.(3.54). and for  $\eta \gtrsim \eta_{re}$  we obtained the approximate form of the solution in eq.(3.49). We connect the solutions at  $\eta = \eta_{re}$  by matching them and their first derivatives,  $\psi_k^{out}(\eta_{re}) = \psi_k^{massive}(\eta_{re})$  and  $\psi_k^{out}(\eta_{re})' = \psi_k^{massive}(\eta_{re})'$ . Using the relations  $H(\eta_{re}) = Ma(\eta_{re})$  and  $\eta_{re}^{-1} \equiv k_{re} = (Ma_1 k_1)^{1/2}$ , the results of the matching give

$$|\beta_k| = \frac{|\frac{1}{2}+m+i|}{\sqrt{\pi}} 2^{n_s-2} |\Gamma(n_s)| z_S^{2m} \left(\frac{g_S}{g_1}\right)^{-l} \left(\frac{k}{k_s}\right)^{-n_s} \left(\frac{k_{re}}{k_s}\right)^{n_1}. \quad (3.58)$$

Since  $k_{re}/k_s = (Ma_1/k_1)^{1/2} z_S$  and since we assume as in [21]  $k_1/a_1 = M_s$  we obtain

$$|\beta_k| = \frac{|\frac{1}{2}+m+i|}{\sqrt{\pi}} 2^{n_s-2} |\Gamma(n_s)| z_S^{m+1/2} \left(\frac{g_S}{g_1}\right)^{-l} \left(\frac{\omega}{\omega_s}\right)^{-n_s} \left(\frac{M}{M_s}\right)^{n_1/2}. \quad (3.59)$$

Using eq.(3.31), and the definition of  $n_1$ , we obtain the final result

$$\left[ \frac{d\Omega_\psi}{d\ln\omega} \right]_{n_1<1} = \frac{|\frac{1}{2}+m+i|^2}{\pi} 2^{2n_s-4} \Gamma^2(n_s) z_{eq}^{-1} g_1^2 z_S^{2m-3} \left(\frac{g_S}{g_1}\right)^{-2l} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^{3-2n_s} \left(\frac{M}{M_s}\right)^{1/2-m}. \quad (3.60)$$

Note that the spectrum is actually partially amplified because the relative density of massive particles during RD is growing as the scale factor.

## 2. Massive spectrum: The contribution of the time dependent mode of $\psi$

As in the massless approximation, we assume that  $H_1^2 > k^2 + M^2 a_1^2$ , for all  $k$ . We use our method of calculating the contribution of the time dependent mode of  $\psi$  by using its correspondence to the constant mode of  $\Pi$ . The solutions are given in eq.(3.28). We reproduce here their approximate forms,

$$\Pi_k^{out} = S(\eta) \sqrt{\pi} 2^{n_1 - n_s - 1} \frac{\Gamma(-n_s^\pi)}{\Gamma(-n_1^\pi)} z_S^{-2m} \left(\frac{g_S}{g_1}\right)^l \left(\frac{k}{k_s}\right)^{-n_1 + n_s} k \sqrt{\eta} J_{n_1^\pi}(k\eta) \text{ for } n_1 > 0 \quad (3.61)$$

$$\Pi_k^{out} = S(\eta) \frac{\sqrt{\pi} 2^{3n_1 - n_s - 2}}{n_1(1 - 2n_1)} \frac{\Gamma(-n_s^\pi)}{\Gamma(n_1^\pi)} z_S^{-1} \left(\frac{g_S}{g_1}\right)^l \left(\frac{k}{k_s}\right)^{n_1 + n_s} k \sqrt{\eta} J_{-n_1^\pi}(k\eta) \text{ for } n_1 < 0. \quad (3.62)$$

In the case of massive particles these solutions are correct only for  $H^2 > k^2 + M^2 a^2$ . Therefore for  $k\eta < 1$  (and  $M < k/a$ ) the leading terms are

$$\Pi_k^{out} = \sqrt{\pi k_s} 2^{1 - n_s^\pi} \Gamma(-n_s^\pi) a_s^m g_S^l \left(\frac{k}{k_s}\right)^{n_s} \text{ for } n_1 > 0, \quad (3.63)$$

and

$$\Pi_k^{out} = \sqrt{\pi k_s} \frac{2^{4n_1 - n_s - 3}}{n_1(1 - 2n_1)} \Gamma(-n_s^\pi) a_s^m g_S^l z_S^{-1 + 2m} \left(\frac{k}{k_s}\right)^{1 + n_s - 2m} (k\eta)^{2m+1} \text{ for } n_1 < 0. \quad (3.64)$$

(a) *The case  $n_1 > 0$ .*

In the case  $n_1 > 0$ , the approximate solution for  $\eta \lesssim \eta_{re}$  is given in eq.(3.63) and for  $\eta \gtrsim \eta_{re}$  it is given in eq.(3.50). We connect the solutions at  $\eta = \eta_{re}$  by matching them and their first derivatives,  $\Pi_k^{out}(\eta_{re}) = \Pi_k^{massive}(\eta_{re})$  and  $\Pi_k^{out}(\eta_{re})' = \Pi_k^{massive}(\eta_{re})'$ , and obtain

$$|\beta_k| = \left|\frac{1}{2} + m + i\right| 2^{-2 - n_s^\pi} \sqrt{\pi} |\Gamma(-n_s^\pi)| z_S^{-2m} \left(\frac{g_S}{g_1}\right)^l \left(\frac{k}{k_s}\right)^{n_s} \left(\frac{k_{re}}{k_s}\right)^{-n_1} \quad (3.65)$$

Using  $k_{re}/k_s = (Ma_1/k_1)^{1/2} z_S$  and since we assume as in [21]  $k_1/a_1 = M_s$ ,

$$|\beta_k| = \left|\frac{1}{2} + m + i\right| 2^{-2 - n_s^\pi} \sqrt{\pi} |\Gamma(-n_s^\pi)| z_S^{-m-1/2} \left(\frac{g_S}{g_1}\right)^l \left(\frac{\omega}{\omega_s}\right)^{n_s} \left(\frac{M}{M_s}\right)^{-n_1/2} \quad (3.66)$$

Using eq.(3.31), and the definitions of  $n_s^\pi$  and  $n_1$ , we get the final result

$$\left[\frac{d\Omega_\pi}{d\ln\omega}\right]_{n_1 > 0} = \left|\frac{1}{2} + m + i\right|^2 \frac{\pi}{2^{2+2n_s}} \Gamma^2(1 - n_s) z_{eq}^{-1} g_1^2 z_S^{-2m-5} \left(\frac{g_S}{g_1}\right)^{2l} \frac{\sqrt{M^2 + \omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^{3+2n_s} \left(\frac{M}{M_s}\right)^{m-1/2} \quad (3.67)$$

(b) *The case  $n_1 < 0$ .*

In the case  $n_1 < 0$ , the approximate solution for  $\eta \lesssim \eta_{re}$  is given in eq.(3.64) and for  $\eta \gtrsim \eta_{re}$  it is given in eq.(3.50). We connect the solutions at  $\eta = \eta_{re}$  by matching them and their first derivatives,  $\Pi_k^{out}(\eta_{re}) = \Pi_k^{massive}(\eta_{re})$  and  $\Pi_k^{out}(\eta_{re})' = \Pi_k^{massive}(\eta_{re})'$ . Using  $k_{re}/k_s = (Ma_1/k_s)^{1/2} z_S$ , and since we assume as in [21]  $k_1/a_1 = M_s$ , we obtain

$$|\beta_k| = \left|-\frac{1}{2} - m + i\right| \frac{\sqrt{\pi} 2^{4n_1 - n_s - 4}}{n_1(1 - 2n_1)} |\Gamma(-n_s^\pi)| z_S^{-5/2 - m} \left(\frac{g_S}{g_1}\right)^l \left(\frac{\omega}{\omega_s}\right)^{2+n_s} \left(\frac{M}{M_s}\right)^{n_1/2 - 1} \quad (3.68)$$

Using eq.(3.31), and the definitions of  $n_s^\pi$  and  $n_1$ , we obtain the final result

$$\begin{aligned} \left[\frac{d\Omega_\pi}{d\ln\omega}\right]_{n_1 < 0} &= \left|-\frac{1}{2} - m + i\right|^2 \pi \frac{2^{-6-2n_s-8m}}{m^2(\frac{1}{2} - m)^2} \Gamma^2(1 - n_s) \times \\ & z_{eq}^{-1} g_1^2 z_S^{-9-2m} \left(\frac{g_S}{g_1}\right)^{2l} \frac{\sqrt{M^2 + \omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^{7+2n_s} \left(\frac{M}{M_s}\right)^{-m-3/2} \end{aligned} \quad (3.69)$$

#### IV. SPECTRA OF PRODUCED PARTICLES

In this section we collect the different contributions that we calculated in the previous section and put them together into complete spectra for massless and massive particles. We use this opportunity to recall the parameters that appear in the expressions for the spectra. The spectra we obtain depend on two parameters, the total redshift during the string phase  $z_S = a_1/a_S$ , and the ratio of string coupling at the beginning and end of string phase  $g_S/g_1$ . The coupling  $g_1$  is considered a known number, the value of the string coupling today, and so is the string mass  $M_s$ . The frequency  $\omega_s$  is expressed as  $\omega_s = w_1/z_S$  where  $w_1$  is taken to be known [21]. Also appearing are the redshift since matter-radiation equality  $z_{eq}$ . We recall also the definition,  $n_s = [\sqrt{3}(l+m) + 1 - m]/2$ . All spectra are valid for  $\omega \leq \omega_s$ , unless otherwise stated. We present the resulting spectra in their original form, but also in a more symmetric form in the spirit of [18], to help in the possible uncovering and understanding of some underlying symmetry.

##### A. The spectrum of massless particles

(a) *The case  $n_1 > 1$  ( $m < -\frac{1}{2}$ ).*

In this case  $\frac{d\Omega}{d\ln\omega} = \left[\frac{d\Omega_\psi}{d\ln\omega}\right]_{n_1>1} + \left[\frac{d\Omega_\pi}{d\ln\omega}\right]_{n_1>0}$ . The relevant expressions appear in eq.(3.36) and (3.40).

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} = & \frac{2^{2n_s-2m-1}}{(\frac{1}{4}-m^2)^2} \frac{\Gamma^2(n_s)}{\Gamma^2(m-\frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-6} \left(\frac{g_S}{g_1}\right)^{-2l} \left(\frac{\omega}{\omega_s}\right)^{7+2m-2n_s} \\ & + 2^{1-2m-2n_s} \frac{\Gamma^2(1-n_s)}{\Gamma^2(m+\frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-4m-4} \left(\frac{g_S}{g_1}\right)^{2l} \left(\frac{\omega}{\omega_s}\right)^{3+2m+2n_s}. \end{aligned} \quad (4.1)$$

The leading contribution is determined by the ratio of the two terms which is approximately  $z_S^{2-4m}(\frac{g_S}{g_1})^{4l}$ , if  $\frac{g_S}{g_1} \lesssim 1$  and  $z_S \gg 1$  then the second term will dominate. The frequency dependence of the second (and leading) term in eq.(4.1) is determined by the index  $n = 3 + 2m + 2n_s = 4 + m + \sqrt{3}(l+m)$ . So, for example, for a perturbative dilaton dependence  $l = -1$ ,  $n = 4 - \sqrt{3} + (1 + \sqrt{3})m$ . Therefore, all spectra with  $l = -1$  and  $m \leq -1$  will be decreasing spectra. In particular, the case  $l = -1$  and  $m = -1$  corresponding to the usual antisymmetric tensor will have  $n = 3 - 2\sqrt{3} \simeq -0.46$ . More negative values of  $m$  will give more sharply decreasing spectra.

We may present eq.(4.1) in a more symmetric form suggested in [18]. We do this to allow for a future study to reveal underlying symmetry.

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} = & \mathcal{N}_{l,m}^I z_{eq}^{-1} g_1^2 z_S^{-2m-5} \left(\frac{\omega}{\omega_s}\right)^{5+2m} \\ & \times \left\{ \mathcal{A}_{l,m}^I z_S^{2m-1} \left(\frac{g_S}{g_1}\right)^{-2l} \left(\frac{\omega}{\omega_s}\right)^{2-2n_s} + \mathcal{A}_{l,m}^{I-1} z_S^{-2m+1} \left(\frac{g_S}{g_1}\right)^{2l} \left(\frac{\omega}{\omega_s}\right)^{-2+2n_s} \right\}, \end{aligned} \quad (4.2)$$

where  $\mathcal{N}_{l,m}^{II}$  and  $\mathcal{A}_{l,m}^{II}$  can be read off eq.(4.1).

(b) *The case  $1 > n_1 > 0$  ( $\frac{1}{2} > m > -\frac{1}{2}$ ).*

In this case  $\frac{d\Omega}{d\ln\omega} = \left[\frac{d\Omega_\psi}{d\ln\omega}\right]_{n_1<1} + \left[\frac{d\Omega_\pi}{d\ln\omega}\right]_{n_1>0}$ . The relevant expressions are given in eq.(3.35) and (3.40).

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} = & 2^{2n_s+2m-1} \frac{\Gamma^2(n_s)}{\Gamma^2(\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{4m-4} \left(\frac{g_S}{g_1}\right)^{-2l} \left(\frac{\omega}{\omega_s}\right)^{5-2m-2n_s} \\ & + 2^{1-2m-2n_s} \frac{\Gamma^2(1-n_s)}{\Gamma^2(m+\frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-4m-4} \left(\frac{g_S}{g_1}\right)^{2l} \left(\frac{\omega}{\omega_s}\right)^{3+2m+2n_s}. \end{aligned} \quad (4.3)$$

The leading contribution is determined by the ratio of the two terms which is approximately  $z_S^{-8m}(\frac{g_S}{g_1})^{4l}$ , if  $\frac{g_S}{g_1} \lesssim 1$  and  $z_S \gg 1$  then the dominant term is determined by the sign of  $m$ . The case  $m = 0$ ,  $l = -1$  corresponds to photons, studied in [12]. In that case, the second term dominates because of the factor  $(g_1/g_S)^4$ . It is the only term computed in [12]. The index determining the frequency dependence is  $n = 4 - \sqrt{3}$ . It is comforting to observe that our method of calculation, not relying on specific background evolution during the string phase gives the same results (where available) as a calculation relying on some specific background evolution.

We may represent eq.(4.3) in a more symmetric form. We do this to allow for a future study to reveal underlying symmetry.

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} &= \mathcal{N}_{l,m}^{II} z_{eq}^{-1} z_S^{-4} g_1^2 \left( \frac{\omega}{\omega_s} \right)^4 \\ &\times \left\{ \mathcal{A}_{l,m}^{II} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{1-2m-2n_s} + \mathcal{A}_{l,m}^{II-1} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{-1+2m+2n_s} \right\}, \end{aligned} \quad (4.4)$$

where  $\mathcal{N}_{l,m}^{II}$  and  $\mathcal{A}_{l,m}^{II}$  can be read off eq.(4.3).

(c) *The case  $n_1 < 0$  ( $m > \frac{1}{2}$ ).*

In this case  $\frac{d\Omega}{d\ln\omega} = \left[ \frac{d\Omega_\psi}{d\ln\omega} \right]_{n_1 < 1} + \left[ \frac{d\Omega_\pi}{d\ln\omega} \right]_{n_1 < 0}$ . The relevant expressions are given in eq.(3.35) and (3.41).

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} &= 2^{2n_s+2m-1} \frac{\Gamma^2(n_s)}{\Gamma^2(\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{4m-4} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{5-2m-2n_s} \\ &+ \frac{2^{-6m-2n_s-5}}{m^2(m-\frac{1}{2})^2} \frac{\Gamma^2(1-n_s)}{\Gamma^2(-\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{-6} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{5-2m+2n_s}. \end{aligned} \quad (4.5)$$

The leading contribution is determined by the ratio of the two terms which is approximately  $z_S^{-4m-2}(\frac{g_S}{g_1})^{4l}$ , if  $\frac{g_S}{g_1} \lesssim 1$  and  $z_S \gg 1$  then the first term will dominate. The frequency dependence of the first (and leading) term in eq.(4.5) is determined by the index  $n = 5 - 2m - 2n_s = 4 - m - \sqrt{3}(l+m)$ . So, for example, for a perturbative dilaton dependence  $l = -1$ ,  $n = 4 + \sqrt{3} - (1 + \sqrt{3})m$ . Therefore, all spectra with  $l = -1$  and  $m \leq 1$  will be increasing spectra. In particular, the case  $l = -1$  and  $m = 1$  corresponding to gravitons and dilatons will have  $n = 3$  as in [17,5] (we have used an approximation in which the logarithmic factors are missing). Again, we observe that our method of calculation, not relying on specific background evolution during the string phase gives the same results (where available) as a calculation relying on some specific background evolution.

More negative values of  $m$  will correspondingly give more sharply increasing spectra. The spectrum of the model independent axion, for which  $l = 1$ ,  $m = 1$ , will have an index  $n = 4 - 2\sqrt{3} \simeq -0.46$ . Note that this is exactly the index of the leading contribution to the spectrum of the antisymmetric tensor (a fact known to the authors of [19]). In this class of generically increasing spectra the axion is standing out with it's decreasing spectrum.

We may represent eq.(4.5) in a more symmetric form. We do this to allow for a future study to reveal underlying symmetry.

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} &= \mathcal{N}_{l,m}^{III} z_{eq}^{-1} z_S^{2m-5} g_1^2 \left( \frac{\omega}{\omega_s} \right)^{5-2m} \\ &\times \left\{ \mathcal{A}_{l,m}^{III} z_S^{2m+1} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{-2n_s} + \mathcal{A}_{l,m}^{III-1} z_S^{-2m-1} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{2n_s} \right\}, \end{aligned} \quad (4.6)$$

where  $\mathcal{N}_{l,m}^{III}$  and  $\mathcal{A}_{l,m}^{III}$  can be read off eq.(4.5).

## B. The spectrum of massive particles

### 1. The discontinuity in the spectrum of the massive particles

The spectrum we computed in section (3.C) was that of massive particles that enter the horizon as massive particles, so at  $\eta_{re}$ ,  $k/a(\eta_{re}) < M$ . The spectrum of massive particles computed in section (3.B), is valid for massive particles if at  $\eta_{re}$ ,  $k/a(\eta_{re}) > M$ . In the approximation scheme that we are using, there will be a discontinuity in the slope of the spectrum of massive particles, at a wavenumber  $k_m$ , where  $k_m$  is defined by the condition  $k_m/a(\eta_m) = M$ . The moment  $\eta_m$  is the (conformal) time where the approximation  $k/a(\eta_m) > M$  at reentry is no longer correct. To compute  $k_m$  we switch momentarily to cosmic time. First, since  $t_m$  is during RD (see below), it can be computed from  $M = H(t_m) = H_1 t_1/t_m$ . Since  $a(t_m)$  depends on whether  $t_m$  is bigger or smaller then  $t_{eq}$ , we get that if  $t_m < t_{eq}$  (i.e.  $M > H(t_{eq}) = 10^{-27}ev$ ), all the particles will get into the horizon in the RD period. Therefore



$a(t_m) = a_1(t_m/t_1)^{1/2} = a_1(H_1/M)^{1/2}$  and since  $H_1 = k_1/a_1$  we obtain  $a(t_m) = (k_1 a_1/M)^{1/2}$ , so  $k_m = (M a_1 k_1)^{1/2}$ . Because  $\frac{\omega_m}{\omega_1} = \frac{k_m}{k_1}$  and  $k_1/a_1 = M_s$ ,  $\frac{k_m}{k_1} = \frac{(M a_1 k_1)^{1/2}}{M_s a_1}$ . Therefore

$$\frac{k_m}{k_1} = \left( \frac{M}{M_s} \right)^{1/2} \quad (4.7)$$

The discontinuity of the slope will be at a frequency  $\omega_m = \omega_1 \left( \frac{M}{M_s} \right)^{1/2}$ . At that frequency

$$\frac{\omega_m}{\omega_s} = z_S \left( \frac{M}{M_s} \right)^{1/2} \quad (4.8)$$

If  $\omega_s > M$ , there will be an additional change in the slope at  $\omega = M$ , above which  $\frac{\sqrt{M^2 + \omega^2}}{\omega_s} \simeq \frac{\omega}{\omega_s}$ , and the spectrum of massive particles and massless particles become approximately the same.

## 2. Spectrum of massive particles

(a) The case  $n_1 > 1$  ( $m < -\frac{1}{2}$ ).

In this case  $\frac{d\Omega}{d \ln \omega} = \left[ \frac{d\Omega_\psi}{d \ln \omega} \right]_{n_1 > 1} + \left[ \frac{d\Omega_\pi}{d \ln \omega} \right]_{n_1 > 0}$ .

For  $\omega_m < \omega < \omega_s$  the relevant expressions are given in eq.(3.43) and (3.44).

$$\begin{aligned} \frac{d\Omega}{d \ln \omega} &= \frac{2^{2n_s-2m-1}}{(\frac{1}{4} - m^2)^2} \frac{\Gamma^2(n_s)}{\Gamma^2(m - \frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-6} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{6+2m-2n_s} \frac{\sqrt{M^2 + \omega^2}}{\omega_s} \\ &+ 2^{1-2m-2n_s} \frac{\Gamma^2(1-n_s)}{\Gamma^2(m + \frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-4m-4} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{2+2m+2n_s} \frac{\sqrt{M^2 + \omega^2}}{\omega_s}. \end{aligned} \quad (4.9)$$

For  $\omega < \omega_m$ , the relevant expressions are given in eq.(3.57) and (3.67).

$$\begin{aligned} \frac{d\Omega}{d \ln \omega} &= \frac{|3m+i|^2}{\pi} \frac{2^{2n_s-6}}{(\frac{1}{4} - m^2)^2} \Gamma^2(n_s) z_{eq}^{-1} g_1^2 z_S^{2m-7} \left( \frac{g_S}{g_1} \right)^{-2l} \left( \frac{\omega}{\omega_s} \right)^{7-2n_s} \frac{\sqrt{M^2 + \omega^2}}{\omega_s} \left( \frac{M}{M_s} \right)^{m-1/2} \\ &+ \left| \frac{1}{2} + m + i \right|^2 \frac{\pi}{2^{2+2n_s}} \Gamma^2(1-n_s) z_{eq}^{-1} g_1^2 z_S^{-2m-5} \left( \frac{g_S}{g_1} \right)^{2l} \left( \frac{\omega}{\omega_s} \right)^{3+2n_s} \frac{\sqrt{M^2 + \omega^2}}{\omega_s} \left( \frac{M}{M_s} \right)^{m-1/2}. \end{aligned} \quad (4.10)$$

At  $\omega = \omega_m$  we may check, using eq.(4.8) that the spectrum is continuous, up to numerical factors of order 1. The spectral index, however, jumps at  $\omega_m$ .

If  $\omega_s > M$ , then for the range  $M < \omega < \omega_s$  the spectrum is approximately the same as the massless spectrum, eq.(4.1).

As in the massless case we may determine the leading contribution to the spectrum. It is determined by the ratio of the terms which is approximately  $z_S^{2-4m} (\frac{g_S}{g_1})^{4l}$ , if  $\frac{g_S}{g_1} \lesssim 1$  and  $z_S \gg 1$  then the second term will dominate. For example, for  $l = -1$  and  $m = -1$  the spectral index will be  $n = 2 - 2\sqrt{3} \simeq -1.46$  for  $w > \omega_m$  and  $n = 5 - 2\sqrt{3} \simeq +1.53$  for  $w < \omega_m$ . If  $\omega_s > M$ , then for  $M < \omega < \omega_s$  the index is  $n = 3 - 2\sqrt{3} \simeq -0.46$ .

Eqs.(4.9), (4.10) may be presented in a more symmetric, perhaps related to some symmetry of the underlying physics. We present here only the symmetric form of eq.(4.10).

$$\begin{aligned} \frac{d\Omega}{d \ln \omega} &= \mathcal{M}_{l,m}^I z_{eq}^{-1} g_1^2 z_S^{-6} \frac{\sqrt{M^2 + \omega^2}}{\omega_s} \left( \frac{\omega}{\omega_s} \right)^5 \left( \frac{M}{M_s} \right)^{m-\frac{1}{2}} \\ &\times \left\{ \mathcal{B}_{l,m}^I \left( \frac{g_S}{g_1} \right)^{-2l} z_S^{2m-1} \left( \frac{\omega}{\omega_s} \right)^{2-2n_s} + \mathcal{B}_{l,m}^{I-1} \left( \frac{g_S}{g_1} \right)^{2l} z_S^{-2m+1} \left( \frac{\omega}{\omega_s} \right)^{-2+2n_s} \right\}, \end{aligned} \quad (4.11)$$

where  $\mathcal{M}_{l,m}^I$  and  $\mathcal{B}_{l,m}^I$  are numerical coefficients of order one which could be read off eq.(4.10).

(b) The case  $1 > n_1 > 0$  ( $-\frac{1}{2} < m < \frac{1}{2}$ ).

In this case  $\frac{d\Omega}{d \ln \omega} = \left[ \frac{d\Omega_\psi}{d \ln \omega} \right]_{n_1 < 1} + \left[ \frac{d\Omega_\pi}{d \ln \omega} \right]_{n_1 > 0}$ .

For  $\omega_m < \omega < \omega_s$  the relevant expressions are given in eq.(3.42) and (3.44).

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} &= 2^{2n_s+2m-1} \frac{\Gamma^2(n_s)}{\Gamma^2(\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{4m-4} \left(\frac{g_S}{g_1}\right)^{-2l} \left(\frac{\omega}{\omega_s}\right)^{4-2m-2n_s} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \\ &+ 2^{1-2m-2n_s} \frac{\Gamma^2(1-n_s)}{\Gamma^2(m+\frac{1}{2})} z_{eq}^{-1} g_1^2 z_S^{-4m-4} \left(\frac{g_S}{g_1}\right)^{2l} \left(\frac{\omega}{\omega_s}\right)^{2+2m+2n_s} \frac{\sqrt{M^2+\omega^2}}{\omega_s}. \end{aligned} \quad (4.12)$$

For  $\omega < \omega_m$ , the relevant expressions are given in eq.(3.60) and (3.67).

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} &= \frac{|\frac{1}{2}+m+i|^2}{\pi} 2^{2n_s-4} \Gamma^2(n_s) z_{eq}^{-1} g_1^2 z_S^{2m-3} \left(\frac{g_S}{g_1}\right)^{-2l} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^{3-2n_s} \left(\frac{M}{M_s}\right)^{1/2-m} \\ &+ \left|\frac{1}{2}+m+i\right|^2 \frac{\pi}{2^{2+2n_s}} \Gamma^2(1-n_s) z_{eq}^{-1} g_1^2 z_S^{-2m-5} \left(\frac{g_S}{g_1}\right)^{2l} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^{3+2n_s} \left(\frac{M}{M_s}\right)^{m-1/2}. \end{aligned} \quad (4.13)$$

At  $\omega = \omega_m$  we may check, using eq.(4.8) that the spectrum is continuous, up to numerical factors of order 1.

If  $\omega_s > M$ , then for the range  $M < \omega < \omega_s$  the spectrum is approximately the same as the massless spectrum, eq.(4.3).

The determination of the leading contribution is more complicated now, because the ratio of the two terms in each range is different and involves also the ratio  $\frac{M}{M_s}$ . We postpone such analysis to a more detailed study of specific cases.

Eqs.(4.12), (4.13) may be presented in a more symmetric, perhaps related to some symmetry of the underlying physics. We present here only the symmetric form of eq.(4.13).

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} &= \mathcal{M}_{l,m}^{II} z_{eq}^{-1} z_S^{-4} g_1^2 \frac{\sqrt{M^2+\omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^3 \\ &\times \left\{ \mathcal{B}_{l,m}^{II} z_S^{2m+1} \left(\frac{g_S}{g_1}\right)^{-2l} \left(\frac{\omega}{\omega_s}\right)^{-2n_s} \left(\frac{M}{M_s}\right)^{m-\frac{1}{2}} + \mathcal{B}_{l,m}^{II} z_S^{-1-2m} \left(\frac{g_S}{g_1}\right)^{2l} \left(\frac{\omega}{\omega_s}\right)^{2n_s} \left(\frac{M}{M_s}\right)^{\frac{1}{2}-m} \right\}, \end{aligned} \quad (4.14)$$

where  $\mathcal{M}_{l,m}^{II}$  and  $\mathcal{B}_{l,m}^{II}$  are numerical coefficients of order one which could be read off eq.(4.13).

(c) The case  $n_1 < 0$  ( $m > \frac{1}{2}$ ).

In this case  $\frac{d\Omega}{d\ln\omega} = \left[\frac{d\Omega_\psi}{d\ln\omega}\right]_{n_1 < 1} + \left[\frac{d\Omega_\pi}{d\ln\omega}\right]_{n_1 < 0}$ .

For  $\omega_m < \omega < \omega_s$  the relevant expressions are given in eq.(3.42) and (3.45).

$$\begin{aligned} \frac{d\Omega}{d\ln\omega} &= 2^{2n_s+2m-1} \frac{\Gamma^2(n_s)}{\Gamma^2(\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{4m-4} \left(\frac{g_S}{g_1}\right)^{-2l} \left(\frac{\omega}{\omega_s}\right)^{4-2m-2n_s} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \\ &+ \frac{2^{-6m-2n_s-5}}{m^2(m-\frac{1}{2})^2} \frac{\Gamma^2(1-n_s)}{\Gamma^2(-\frac{1}{2}-m)} z_{eq}^{-1} g_1^2 z_S^{-6} \left(\frac{g_S}{g_1}\right)^{2l} \left(\frac{\omega}{\omega_s}\right)^{4-2m+2n_s} \frac{\sqrt{M^2+\omega^2}}{\omega_s}. \end{aligned} \quad (4.15)$$

For  $\omega < \omega_m$ , the relevant expressions are given in eq.(3.60) and (3.69).

$$\begin{aligned} \left[\frac{d\Omega_\psi}{d\ln\omega}\right] &= \frac{|\frac{1}{2}+m+i|^2}{\pi} 2^{2n_s-4} \Gamma^2(n_s) z_{eq}^{-1} g_1^2 z_S^{2m-3} \left(\frac{g_S}{g_1}\right)^{-2l} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^{3-2n_s} \left(\frac{M}{M_s}\right)^{1/2-m} + \\ &|\frac{1}{2}+m-i|^2 \pi \frac{2^{-6-2n_s-8m}}{m^2(\frac{1}{2}-m)^2} \Gamma^2(1-n_s) z_{eq}^{-1} g_1^2 z_S^{-9-2m} \left(\frac{g_S}{g_1}\right)^{2l} \frac{\sqrt{M^2+\omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^{7+2n_s} \left(\frac{M}{M_s}\right)^{-m-3/2}. \end{aligned} \quad (4.16)$$

At  $\omega = \omega_m$  we may check, using eq.(4.8) that the spectrum is continuous, up to numerical factors of order 1.

If  $\omega_s > M$ , then for the range  $M < \omega < \omega_s$  the spectrum is approximately the same as the massless spectrum, eq.(4.5).

The determination of the leading contribution is more complicated now, because the ratio of the two terms in each range is different and involves also the ratio  $\frac{M}{M_s}$ . We postpone such analysis to a more detailed study of specific cases.

Eqs.(4.15), (4.16) may be presented in a more symmetric, perhaps related to some symmetry of the underlying physics. We present here only the symmetric form of eq.(4.16).

$$\frac{d\Omega}{d\ln\omega} = \mathcal{M}_{l,m}^{III} z_{eq}^{-1} z_S^{-6} g_1^2 \frac{\sqrt{M^2 + \omega^2}}{\omega_s} \left(\frac{\omega}{\omega_s}\right)^5 \left(\frac{M}{M_s}\right)^{-\frac{1}{2}-m} \times \left\{ \mathcal{B}_{l,m}^{III} z_S^{2m+3} \left(\frac{g_S}{g_1}\right)^{-2l} \left(\frac{\omega}{\omega_s}\right)^{-2-2n_s} \left(\frac{M}{M_s}\right) + \mathcal{B}_{l,m}^{III-1} z_S^{-3-2m} \left(\frac{g_S}{g_1}\right)^{2l} \left(\frac{\omega}{\omega_s}\right)^{2+2n_s} \left(\frac{M}{M_s}\right)^{-1} \right\}, \quad (4.17)$$

where  $\mathcal{M}_{l,m}^{III}$  and  $\mathcal{B}_{l,m}^{III}$  are numerical coefficients of order one which could be read off eq.(4.16).

The spectra of massless and massive particles can be translated into each other by using the relations  $M/M_s = (k_{re}/k_s)^2 z_S^{-2}$  for the massive case and  $(k_{re}/k_s)^2 z_S^{-2} = (\omega/\omega_s)^2 z_S^{-2}$  for the massless case. The substitutions  $(\omega/\omega_s)^2 z_S^{-2} \leftrightarrow M/M_s$  in addition to the substitution accounting for the difference in the energy  $\sqrt{M^2 + \omega^2}/\omega_s \leftrightarrow \omega/\omega_s$  translate the results from the massive case into those of the massless case and vice versa.

## V. CONCLUSIONS

Spectra of produced particles in string cosmology models come in different shapes, as summarized in section 4. Even though the background curvature is increasing during the inflationary dilaton-driven phase, spectra are not necessarily increasing. This peculiarity comes about because different types of particles couple differently to the background curvature and dilaton, and in addition, spectra of massless and massive particles are different.

During the string phase, the use of the lowest order equations for the background or for perturbations is questionable, and therefore it might have been possible to doubt the validity and accuracy of the calculations of particle production. We have argued that to estimate particle production during the dilaton-driven phase it is not necessary to know the details of the evolution during the string phase. If the principle of causality is used, expressed in practical terms by the “freezing” of the perturbation amplitude and its conjugate momentum, it is possible to do reliable calculations. Our results provide several explicit checks of this principle, by comparison with other calculations, and therefore give further credibility to existing spectral calculations.

We have not used any explicit symmetry considerations in our calculations. However, it is clear that spectral indices and perhaps other properties of the spectra could be analyzed using an underlying symmetry as in [18].

Among the spectra we found there are some with weak dependence on frequency and some decreasing spectra which contain substantial power at large wavelength. Because these spectra depend differently on the basic parameters of the models, requiring that they be compatible with astrophysical and cosmological bounds is likely to narrow the allowed range of the parameters. Their existence also suggests an obvious source for the observed large scale anisotropy.

Relic massive particles are an appealing source for cold or hot dark matter. However, to decide on this issue, further analysis and input are necessary. To compute relic abundances of particles we need to know more about their interactions and also about the late time evolution of the Universe. We are completing such analysis for axions, and hope to perform similar analyses for other particles.

As already mentioned several times, there are obvious improvements and generalizations to our calculations. Straightforward improvements, that cannot however be performed in a general way, concern the late time background evolution, and the inclusion of effects of particle decay and interactions. Among possible generalizations, we view considering other background evolutions as particularly interesting. This was partially done in [14,19] and should provide information about the robustness of the spectra.

## ACKNOWLEDGMENTS

We would like to thank A. Buonanno, K. Meissner, C. Ungarelli and G. Veneziano for sharing with us unpublished results and for useful discussions of several aspects of our calculations. This work is supported in part by the Israel Science Foundation administered by the Israel Academy of Sciences and Humanities.

- 
- [1] G. Veneziano, *Phys. Lett.* B265 (1991) 287.
  - [2] M. Gasperini and G. Veneziano, *Astropart. Phys.* 1 (1993) 317.
  - [3] R. Brustein and G. Veneziano, *Phys. Lett.* B329 (1994) 429.

- [4] N. Kaloper, R. Madden and K.A. Olive, *Nucl. Phys.* B452 (1995) 677.
- [5] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Lett.* B361 (1995) 45.
- [6] G. Veneziano, hep-th/9703150.
- [7] E. Weinberg and M. Turner, hep-th/9705035.
- [8] M. Maggiore and R. Sturani, hep-th/9706053;  
A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, hep-th/9706221.
- [9] M. Gasperini, M. Maggiore and G. Veneziano, *Nucl. Phys.* B494 (1997) 315.
- [10] R. Brustein and R. Madden, hep-th/9702043;  
R. Brustein and R. Madden, hep-th/9708046.
- [11] M. Gasperini and M. Giovannini, *Phys. Lett.* B282 (1992) 36;  
M. Gasperini and M. Giovannini, *Phys. Rev.* D47 (1993) 1519;  
R. Brustein, In Cascina 1996, Gravitational waves pp. 149-152, hep-th/9604159;  
M. Gasperini, hep-th/9607146;  
B. Allen and R. Brustein, *Phys. Rev.* D55 (1997) 3260;  
M. Giovannini, hep-th/9706201.
- [12] M. Gasperini, M. Giovannini and G. Veneziano, *Phys. Rev. Lett.* 75 (1995) 3796;  
D. Lemoine and M. Lemoine, *Phys. Rev.* D52 (1995) 1955.
- [13] M. Gasperini, *Phys. Lett.* B327 (1994) 214;  
M. Gasperini and G. Veneziano, *Phys. Rev.* D50 (1994) 2519.
- [14] E. J. Copeland, R. Easther, D. Wands, *Phys. Rev.* D56 (1997) 874;  
E. J. Copeland, J. E. Lidsey, D. Wands, hep-th/9705050.
- [15] A. Buonanno, M. Maggiore and C. Ungarelli *Phys. Rev.* D55 (1997) 3330;  
M. Galluccio, M. Litterio, F. Occhionero, *Phys. Rev. Lett.* 79 (1997) 970;  
M. Maggiore, *Phys. Rev.* D56 (1997) 1320;  
M. Gasperini, gr-qc/9704045;  
M. Maggiore and R. Sturani, gr-qc/9705082.
- [16] N.D. Birrell and P.C.W. Davies, Quantum fields in curved space, Cambridge University Press 1984.
- [17] R. Brustein, M. Gasperini, M. Giovannini, V. Mukhanov and G. Veneziano, *Phys. Rev.* D51 (1995) 6744;  
M. Gasperini and M. Giovannini, gr-qc/9604002;  
J.-C. Hwang, hep-th/9608041;  
M. Giovannini, *Phys. Rev.* D55 (1997) 595;  
J.-C. Hwang and H. Noh, gr-qc/9612065.
- [18] R. Brustein, M. Gasperini and G. Veneziano, unpublished.
- [19] A. Buonanno, K. A. Meissner, C. Ungarelli and G. Veneziano, to appear.
- [20] R. Brustein and M. Hadad, to appear.
- [21] R. Brustein, M. Gasperini and G. Veneziano, *Phys. Rev.* D55 (1997) 3882.
- [22] M. Giovannini, *Phys. Rev.* D56 (1997) 631.
- [23] M. Gasperini and G. Veneziano, *Mod. Phys. Lett.* A8 (1993) 3701.
- [24] We have ignored a numerical factor in the definition of  $\Omega$ , which in some conventions is  $8\pi/3$ . We ignore this factor throughout.